Chapter 4

Description of the dispersion refraction effect using the space-time ray optic

§ 1. Introduction

In this chapter we begin to describe the dispersion refraction effect using the space-time ray optic, which is one of the main monograph subjects. At the same time, we will try to find out why the standard STRO version cannot describe this effect. As before, we will use KGE (1.7) as an initial wave equation.

STRO is a generalized HF approximation, which holds true if the following conditions are satisfied [16, 31, 40]:

$$\chi = \max\left\{\frac{\tau, \tau_0}{T_P}; \frac{\tau, \tau_0}{T_W}; \frac{\lambda, \lambda_0}{L_P}; \frac{\lambda, \lambda_0}{L_W}\right\} \ll 1. \tag{4.1}$$

Here χ is a small parameter used during asymptotic expansions in the RO method; $\tau \sim 2\pi/\omega$ is the average wave period; $\lambda \sim 2\pi/k$ is the average wavelength; τ_0 and λ_0 are the spatial and temporal dispersion scales, respectively; T_P and L_P are the spatial and temporal scales of variations in the medium parameters, respectively; and T_W and L_W are the spatial and temporal scales of wave field variations, respectively.

The standard version of STRO is based on the eikonal and transport equations; therefore, we will recall below the methods for deriving these equations.

§ 2. Eikonal and transport equations

The eikonal and transport equations are the first two terms of the infinite ray series. To obtain the equations of this series, we will use the standard technique for expanding wave function U in a power series of large parameter υ (see, e.g., [3]).

Having defined $\upsilon = 1/\chi$, we seek the solution U in the form

$$U(\mathbf{r},t) = \exp(i\nu\Psi(\mathbf{r},t)) \sum_{m=0}^{\infty} \frac{A_m(\mathbf{r},t)}{(i\nu)^m}, \tag{4.2}$$

where $\Psi(\mathbf{r}, t)$ and $A_m(\mathbf{r}, t)$ are the phase and amplitude functions of radius vector $\mathbf{r}$ and time t.

By substituting Ansatz (4.2) into the wave equation (1.7), we obtain the series of equations corresponding to the powers of large parameter υ. The terms of the order $O(\upsilon^2)$ vanish if

$$(\nabla\Psi)^2 - \frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2 + \frac{\omega_L^2}{c^2} = 0 \tag{4.3}$$

(here the medium parameter ω_L is taken into account with the order of υ^2 because the properties of the medium are substantial for wave in this case.)

The terms of the order $O(\upsilon)$ vanish if

$$A_0\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}A_0 + 2\nabla\Psi\cdot\nabla A_0 - \frac{2}{c^2}\frac{\partial\Psi}{\partial t}\frac{\partial A_0}{\partial t} = 0. \tag{4.4}$$

For the rest high-order terms ν^{-m}, we get

$$A_{m+1}\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}A_{m+1} + 2\nabla\Psi\cdot\nabla A_{m+1} - \frac{2}{c^2}\frac{\partial\Psi}{\partial t}\frac{\partial A_{m+1}}{\partial t} +$$

$$+\nabla^2 A_m - \frac{1}{c^2}\frac{\partial^2 A_m}{\partial t^2} = 0. \tag{4.5}$$

The eikonal equation (4.3) is the local dispersion equation for a propagation medium. The transport equation (4.4) describes the main component of the amplitude expansion, whereas (4.5) characterizes corrections of the higher order. Only Eqs. (4.3) and (4.4) are usually considered within the scope of generally accepted RO. We will rewrite these equations, defining gradient $\nabla\Psi$ as a local wave vector $\mathbf{k}(\mathbf{r}, t)$ and $-d\Psi/dt$ as a local frequency $\omega(r, t)$:

$$\mathbf{k}^2 - \frac{\omega^2}{c^2} + \frac{\omega_L^2}{c^2} = 0, \tag{4.6}$$

$$\nabla\cdot\mathbf{k}A + \frac{1}{c^2}\frac{\partial\omega}{\partial t}A + 2\mathbf{k}\cdot\nabla A + \frac{2}{c^2}\omega\frac{\partial A}{\partial t} = 0. \tag{4.7}$$

The eikonal equation (4.6) evidently demonstrates that this is the generalization of the dispersion equation (1.10) for an inhomogeneous medium.

Another, less formal, but probably more physically clear and, in our opinion, more preferential method for obtaining ray equations consists in the substitution of Ansatz [11]

$$U(\mathbf{r},t) = A(\mathbf{r},t)\exp\{i\Psi(\mathbf{r},t)\}$$

into the wave equation (1.7) and in the separation of the real and imaginary parts of the resultant complex-valued equation

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - A(\nabla\Psi)^2 + A\frac{1}{c^2}\left(\frac{\partial\Psi}{\partial t}\right)^2 - A\frac{\omega_L^2}{c^2} +$$

$$+i\nabla^2\Psi A - \frac{i}{c^2}\frac{\partial^2\Psi}{\partial t^2}A + i2\nabla\Psi\cdot\nabla A - i\frac{2}{c^2}\frac{\partial\Psi}{\partial t}\frac{\partial A}{\partial t} = 0. \qquad (1.7a)$$

It is evident that the imaginary part of this equation completely coincides with the transport equation (4.4), whereas the real part contains not only the eikonal equation (4.3) but also the second derivatives of the amplitude function $A(\mathbf{r}, t)$. In usual RO these amplitude additions are considered small and are ignored. In the ray series, these additions are implicitly present in the high-order terms of the amplitude expansion (4.5).

The complex-valued equation (1.7a), completely equivalent to the initial wave equation (1.7) for real field functions, is of interest from the physical viewpoint because this equation explicitly indicates that the first derivatives of the phase function $\Psi(r, t)$ and the second derivatives of the amplitude function $A(r, t)$ are interrelated. This equation will subsequently help us to better understand the physical sense of a smoothly inhomogeneous wave field model to be introduced in this chapter.

§ 3. Standard STRO version

There equation specifying space-time rays in STRO [3, 11, 16, 40] is written as

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_g, \qquad (4.8)$$

where $\mathbf{V}_g$ is the group velocity. This is responsible for the following interrelation between the total (d/dt) and partial derivatives $\partial/\partial t$ and ∇:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla. \qquad (4.9)$$

Using the $\mathbf{V}_g = d\omega/dk$ definition of the group velocity, from the local dispersion equation (4.6) we obtain that

$$\mathbf{V}_g = \frac{d\omega}{d\mathbf{k}} = c^2\frac{\mathbf{k}}{\omega}. \qquad (4.10)$$

Differentiating (4.10), we can obtain the $d\mathbf{V}_g/dt$ derivative which describes refraction effects:

$$\frac{d\mathbf{V}_g}{dt} = \frac{c^2}{\omega}\frac{d\mathbf{k}}{dt} - \frac{c^2\mathbf{k}}{\omega^2}\frac{d\omega}{dt}.$$

The differentiation of (4.6) with respect to $\mathbf{r}$ and t gives the relationship between the partial derivatives:

$$(\mathbf{k} \cdot \nabla)\mathbf{k} + \mathbf{k} \times (\nabla \times \mathbf{k}) = \frac{\omega \nabla \omega}{c^2} - \frac{\omega_L \nabla \omega_L}{c^2} \tag{4.11}$$

and

$$\frac{\partial \omega}{\partial t} = \frac{c^2 \mathbf{k}}{\omega} \cdot \frac{\partial \mathbf{k}}{\partial t} + \frac{\omega_L}{\omega}\frac{\partial \omega_L}{\partial t} = -\frac{c^2 \mathbf{k}}{\omega} \cdot \nabla \omega + \frac{\omega_L}{\omega}\frac{\partial \omega_L}{\partial t}. \tag{4.12}$$

We recall that

$$\frac{\partial \mathbf{k}}{\partial t} = \frac{\partial}{\partial t}(\nabla \Psi) = \nabla\left(\frac{\partial \Psi}{\partial t}\right) = -\nabla \omega.$$

According to the standard procedure, $\mathbf{k} \times (\nabla \times \mathbf{k})$ in (4.11) because from $\mathbf{k} = \nabla\Psi$ it follows that the $\mathbf{k}$ field is curlfree: $\nabla \times \mathbf{k} = 0$.

This condition specifies the field model as a locally plane homogeneous and monochromatic wave, where the $\partial k_y/\partial x$ and $\partial k_x/\partial y$ wavevector derivatives are equal.

We should concentrate on the last circumstance because it causes the systematic error of standard STRO (see below).

We now show that the total frequency derivative on the space-time ray is zero for a stationary medium with $\partial \omega_L/\partial t = 0$:

$$\frac{d\omega}{dt} = 0.$$

Indeed, in accordance with (4.9),

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \mathbf{V}_g \cdot \nabla \omega.$$

On the other hand, for a stationary medium, we obtain from (4.10) and (4.12) that

$$\frac{\partial \omega}{\partial t} = -\mathbf{V}_g \cdot \nabla \omega.$$

The expression for the $d\mathbf{k}/dt$ total derivative follows from (4.9) and (4.11):

$$\frac{d\mathbf{k}}{dt} = \frac{\partial \mathbf{k}}{\partial t} + (\mathbf{V}_g \cdot \nabla)\mathbf{k} = -\frac{\omega_L \nabla \omega_L}{\omega}.$$

Hence, the derivative of the group velocity vector can be written as

$$\frac{d\mathbf{V}_g}{dt} = -c^2 \frac{\omega_L \nabla \omega_L}{\omega^2}. \tag{4.13}$$

Thus, in a homogeneous medium ($\nabla\omega_L = 0$),

$$\frac{d\mathbf{V}_g}{dt} = 0;$$

i.e., any refraction effects that change a ray trajectory are absent.

It is clear that a locally-plane homogeneous monochromatic field model does not describe the dispersion refraction effect.

§ 4. New method for deriving ray equations

The main stages in deriving the equations of modified STRO are as follows:

1. The field distribution at an observation point and in the vicinity of this point is generally written in terms of the Taylor series for amplitude and phase. The field model is not specified at this stage; i.e., specific relationships between partial derivatives are not given.
2. The group velocity vector is calculated at an observation point and in the vicinity of this point.
3. The total derivative of the group velocity vector along a ray, which characterizes a change in the vector propagation direction, is generally found based only on general conditions of STRO applicability (4.1).
4. Finally, a particular wave field model is specified.

Here we will use the most general definition of the group velocity vector as a ratio of the average energy flow density $\langle\mathbf{P}\rangle$ during the period of fast oscillations to the average energy density $\langle W\rangle$ during the same period [11, 38, 47, 66, 67]:

$$\mathbf{V}_g = \frac{\langle\mathbf{P}\rangle}{\langle W\rangle}. \tag{4.14}$$

The relation (4.14) between the group velocity and averaged wave energy characteristics was for the first time established in 1877 by Rayleigh and Reynolds.

Expression (4.10), which is used in the STRO standard version, is a particular case suitable only for the model of locally-plane homogeneous monochromatic wave. Using definition (4.14), we can consider more general models, including slightly inhomogeneous waves, remaining within the scope of RO (4.1).

At point $\mathbf{R}_0$, T_0 and in the vicinity of this point, we represent the amplitude A and phase Ψ of the wave field U as a Taylor series in powers of **r** and t:

$$U(\mathbf{R}_0+\mathbf{r},T_0+t)=\Big\{A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+$$

$$+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big\}\exp\Big\{i(\Psi_0+\mathbf{r}\cdot\mathbf{k}+\frac{1}{2}\mathbf{r}(\mathbf{r}\cdot\nabla)\mathbf{k}-\omega t-$$

$$-\frac{1}{2}\frac{\partial\omega}{\partial t}t^2-\mathbf{r}\cdot\nabla\omega t)\Big\}. \qquad (4.15)$$

Note that the values of all field characteristics (**k**, ω, ∇A , etc.) in expression (4.15) are not defined at this stage and are considered only as certain constants before the powers of the **r** and t variables for point $\mathbf{R}_0$, T_0.

From (4.15) we obtain the expressions for field gradient ∇U and time derivative dU/dt.

After the separation of the real part of (4.15), we have

$$\nabla U=\Big\{\nabla A+(\mathbf{r}\cdot\nabla)\nabla A+\nabla\frac{\partial A}{\partial t}t\Big\}\cos\Psi_1-$$

$$-\Big\{A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2A+\frac{\partial A}{\partial t}t+$$

$$+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big\}\{\mathbf{k}+(\mathbf{r}\cdot\nabla)\mathbf{k}-\nabla\omega t\}\sin\Psi_1, \qquad (4.16)$$

$$\frac{\partial U}{\partial t}=\Big\{\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial t^2}t+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)\Big\}\cos\Psi_1+\Big\{A_0+\mathbf{r}\cdot\nabla A+$$

$$+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\Big(\frac{\nabla\partial A}{\partial t}\Big)t\Big\}\times$$

$$\times\{\omega+\frac{\partial\omega}{\partial t}t+\mathbf{r}\cdot\nabla\omega\}\sin\Psi_1. \qquad (4.17)$$

Here phase Ψ_1 is the polynomial of the second order with respect to **r** and t:

$$\Psi_1=\Psi_0+\mathbf{r}\cdot\mathbf{k}+\frac{1}{2}\mathbf{r}\cdot(\mathbf{r}\cdot\nabla)\mathbf{k}-\omega t-\frac{1}{2}\frac{\partial\omega}{\partial t}t^2-\mathbf{r}\cdot\nabla\omega t.$$

The expressions for **P** and W follow from (1.14), (1.15) and (4.16), (4.17):

$$\mathbf{P}=-c^2\Big\{\Big(\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial t^2}t+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)\Big)\cos\Psi_1+\Big(A_0+\mathbf{r}\cdot\nabla A+$$

$$+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big)\Big(\omega+\mathbf{r}\cdot\nabla\omega+$$

$$+\frac{\partial\omega}{\partial t}t\Big)\sin\Psi_1\Big\}\Big\{\Big(\nabla A+(\mathbf{r}\cdot\nabla)\nabla A+\nabla\frac{\partial A}{\partial t}t\Big)\cos\Psi_1-(A_0+\mathbf{r}\cdot\nabla A+$$

$$+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t+$$

$$+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big)(\mathbf{k}+(\mathbf{r}\cdot\nabla)\mathbf{k}-\nabla\omega t)\sin\Psi_1\Big\},$$

$$W=\frac{1}{2}\Big\{\Big[\Big(\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial t^2}t+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)\Big)\cos\Psi_1+$$

$$+\Big(A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big)\times$$

$$\times\Big(\omega+\mathbf{r}\cdot\nabla\omega+\frac{\partial\omega}{\partial t}t\Big)\sin\Psi_1\Big]^2+c^2\Big[\Big(\nabla A+(r\cdot\nabla)\nabla A+\nabla\Big(\frac{\partial A}{\partial t}t\Big)\Big)\times$$

$$\times\cos\Psi_1-\Big(A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t}t^2+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big)\times$$

$$\times(\mathbf{k}+(\mathbf{r}\cdot\nabla)\mathbf{k}-\nabla\omega t)\sin\Psi_1\Big]^2+\Big(\omega_L+\mathbf{r}\cdot\nabla\omega_L+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2\omega_L\Big)^2\times$$

$$\times\Big(A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\Big(\nabla\frac{\partial A}{\partial t}\Big)t\Big)^2\cos^2\Psi_1\Big\}.$$

We now consider a stationary inhomogeneous medium in the HF region $\omega > \omega_L$.

Assume that the conditions of STRO applicability (4.1) are satisfied; i.e.,

$$\frac{\nabla\cdot\mathbf{k}}{|\mathbf{k}|}\sim\frac{|\nabla\times\mathbf{k}|}{|\mathbf{k}|}\sim\frac{|\nabla\omega|}{\omega}\sim\frac{\nabla^2 A}{|\nabla A|}\sim\frac{|\nabla A|}{A}\sim\frac{1}{L_W};\quad\frac{|\nabla\omega_L|}{\omega_L}\sim\frac{1}{L_P};$$

$$\frac{\partial\omega/\partial t}{\omega}\sim\frac{\partial^2 A/\partial t^2}{\partial A/\partial t}\sim\frac{\partial A/\partial t}{A}\sim\frac{1}{T_W};$$

$$\lambda=\frac{2\pi}{|\mathbf{k}|};\quad\tau=\frac{2\pi}{\omega};\quad\frac{\lambda}{L_W}\sim\frac{\lambda}{L_P}\sim\frac{\tau}{T_W}\sim\chi\ll 1.$$

We find the average energy flux $\mathbf{P}$ and density W at point $\mathbf{R}_0$, T_0 and in the vicinity of this point:

$$\langle\mathbf{P}\rangle=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\mathbf{P}(\mathbf{r},t+\xi)\,d\xi;\quad\langle W\rangle=\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}W(\mathbf{r},t+\xi)\,d\xi.$$

By integrating the obtained expressions for **P** and W during the period $\tau = 2\pi/(\omega + \mathbf{r}\cdot\omega)$, we obtain

$$\langle \mathbf{P}\rangle = \frac{c^2}{2\tau}\left\{-\left(\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial t^2}t+\mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)\right)\times\right.$$
$$\times\left(\nabla A+(\mathbf{r}\cdot\nabla)\nabla A+\nabla\frac{\partial A}{\partial t}t\right)+\left(A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A+\right.$$
$$\left.+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}+\mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\times$$
$$\left.\times\left(\omega+\mathbf{r}\cdot\nabla\omega+\frac{\partial\omega}{\partial t}t\right)(\mathbf{k}+(\mathbf{r}\cdot\nabla)\mathbf{k}-\nabla\omega t)\right\} \qquad (4.18)$$

and

$$\langle W\rangle = \frac{1}{4\tau}\left\{\left(\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial t^2}t+\mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)\right)^2+\right.$$
$$\left.+c^2\left(\nabla A+(\mathbf{r}\cdot\nabla)\nabla A+\left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\right\}+\frac{1}{4\tau}\left(A_0+\mathbf{r}\cdot\nabla A+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2\times\right.$$
$$\left.\times A+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2+\mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\left\{\left(\omega+\mathbf{r}\cdot\nabla\omega+\frac{\partial\omega}{\partial t}t\right)^2+\right.$$
$$\left.+c^2(\mathbf{k}+(\mathbf{r}\cdot\nabla)\mathbf{k}-\nabla\omega t)^2+\left(\omega_L+\mathbf{r}\cdot\nabla\omega_L+\frac{1}{2}(\mathbf{r}\cdot\nabla)^2\omega_L\right)^2\right\}. \qquad (4.19)$$

Here we neglected the terms of the highest orders of smallness that appeared during the integration.

The group velocity vector at point $\mathbf{R}_0$, T_0 and in the vicinity of this point is defined by formulas (4.14), (4.18), and (4.19).

At the central point $\mathbf{R}_0$, T_0, this vector has the simplest form:

$$\mathbf{V}_g = c^2\frac{\omega\mathbf{k}-(\partial A/\partial t)\nabla A/A_0^2}{\omega^2+0.5\,\{(\partial A/\partial t)^2+c^2(\nabla A)^2\}/A_0^2}. \qquad (4.20)$$

This formula is the generalization of the well known expression (4.10) for amplitude-modulated waves. If we reject the terms of the order of χ^2, we obtain the formula coinciding with (4.10):

$$\mathbf{V}_g = c^2\frac{\mathbf{k}}{\omega}.$$

We now find the total derivative d/dt along the ray $\mathbf{r} = \int\mathbf{V}_g dt$ governed by the group velocity vector field. Selecting t as an independent variable and taking into account that

$$\frac{dt}{dt}=1;\quad \frac{d\mathbf{r}}{dt}=\mathbf{V}_g=c^2\frac{\mathbf{k}}{\omega},$$

we differentiate the group velocity vector specified by formulas (4.14), (4.18), and (4.19):

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2\omega_L}{\omega^2}\nabla\omega_L - \frac{c^2\omega_L^2}{\omega^3}\nabla\omega + \frac{c^2\omega_L^2}{\omega^4}(\mathbf{k}\cdot\nabla)\mathbf{k} + \frac{c^2\omega_L^3}{\omega^4}\nabla\omega_L. \quad (4.21)$$

Here we rejected the terms of the order of χ^2.

Expression (4.21) contains all factors that lead to a change in the group velocity vector, i.e., causing ordinary and dispersion refraction effects.

We now demonstrate that formula (4.21) contains the STRO standard version as a particular case.

Substituting the relationship (4.11) between partial derivatives (in other words, determining the field model) into (4.21), we obtain the well-known expression for the derivative of the group velocity vector in standard STRO

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2\omega_L}{\omega^2}\nabla\omega_L \quad (4.22)$$

which completely coincides with (4.13).

§ 5. Field models

In contrast to standard STRM, the general expression (4.21) does not impose any restrictions on the relationship between partial derivatives ωL, ω, and $\mathbf{k}$: compare Eqs. (4.11) and (4.12). For example, Eq. (4.21) remains true if vector $\mathbf{k}$ has a non-zero curl component $\nabla\times\mathbf{k}\neq 0$ (certainly, absolute values of the derivatives should be small, so that the condition of RO applicability (4.1) would be satisfied).

We now consider the wave field with a transverse frequency modulation within the scope of standard STRO. At the first stage, we consider only the case of a homogeneous medium (ω_L = const) for simplicity and in order to demonstrate the dispersion refraction effect in the explicit form.

First of all, we recall the explicit expressions for the vector operators:

$$(\mathbf{k}\cdot\nabla)\mathbf{k} = k_x\frac{\partial\mathbf{k}}{\partial x} + k_y\frac{\partial\mathbf{k}}{\partial y} + k_z\frac{\partial\mathbf{k}}{\partial z},$$

$$\nabla\cdot\mathbf{k} = \frac{\partial k_x}{\partial x} + \frac{\partial k_y}{\partial y} + \frac{\partial k_z}{\partial z},$$

$$\nabla\times\mathbf{k} = \left(\frac{\partial k_z}{\partial y} - \frac{\partial k_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial k_x}{\partial z} - \frac{\partial k_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y}\right)\mathbf{e}_z,$$

so that it would be convenient to trace mathematical manipulation.

For the wave propagating along the x axis, the wave function (4.15) at point X_0, Y_0, T_0 and in the X_0+x, Y_0+y, T_0+t vicinity of this point has the following form:

$$U=\left(A_0+\frac{\partial A}{\partial x}x+\frac{\partial A}{\partial t}t\right)\exp\left\{i\left(k_x x+\frac{\partial k_y}{\partial x}xy+\frac{\partial k_x}{\partial y}yx-\right.\right.$$

$$\left.\left.-\ \omega t-\frac{\partial\omega}{\partial y}yt\right)\right\}. \qquad (4.23)$$

Here,

$$k_x=\sqrt{\frac{\omega^2}{c^2}-\frac{\omega_L^2}{c^2}},\quad \frac{\partial k_y}{\partial x}=\frac{\partial k_x}{\partial y}=\frac{\omega}{c^2k_x}\frac{\partial\omega}{\partial y},\quad \frac{\partial A}{\partial x},\quad \frac{\partial A}{\partial t}$$

are the constants characterizing the wave field and its derivatives at the initial point X_0, Y_0, T_0.

In expression (4.23), we specified only central frequency ω and its transverse gradient $\partial\omega/\partial y$. Derivatives $\partial k_y/\partial x$ and $\partial\omega/\partial t=0$ are defined by formulas (4.11) and (4.12), whereas the $\partial k_y/\partial x=\partial k_x/\partial y$ equality follows from the $\nabla\times\mathbf{k}\neq 0$ condition.

The transport equation (4.7) specifies the relationship between the partial derivatives of the amplitude function

$$\frac{\partial A}{\partial t}=-\frac{c^2k_x}{\omega}\frac{\partial A}{\partial x}.$$

It should be noted here that transverse derivative $\partial A/\partial y$ is absent in the transport equation. This means that a transverse linear amplitude modulation does not influence field description in an RO approximation.

Generally speaking, even within the scope of standard STRO, we can consider a wave with an amplitude linearly varying in the transverse direction as a model; however, this wave does not differ from a homogeneous plane wave, which can be verified by direct substitution of the models into the wave equation because the second derivative is zero for the constant and linear function.

Thus, we have defined all wave parameters corresponding to the standard wave field model.

We now estimate the error of this model by substituting the approximate solution (4.23) into the initial wave equation (1.7). As a result of this procedure, we obtain

$$-2A\frac{\omega}{c^2}\frac{\partial\omega}{\partial y}y-4A\left(\frac{\omega^2}{c^4k_x^2}-\frac{1}{c^2}\right)\left(\frac{\partial\omega}{\partial y}\right)^2y^2-A\left(\frac{\partial\omega}{\partial y}\right)^2\left(2\frac{\omega}{c^2k_x}x-t\right)^2+$$

$$+2i\frac{\partial A}{\partial x}\left(2\frac{\omega}{c^2k_x}-\frac{k_x}{\omega}\right)\frac{\partial\omega}{\partial y}y=0. \qquad (4.24)$$

The maximal discrepancy in the vicinity of point X_0, Y_0, T_0 corresponds to the linear term with y in the real part of (4.24), and this means that the selected field model takes into account not all effects, the spatial scales of which correspond to those of refraction effects. Note, that another linear term that appeared in the imaginary part of (4.24) is of a much higher order of smallness because $dA/dx \ll A\omega/c^2$.

The linear discrepancy term appears because the value of

$$\frac{\omega}{c^2 k_x}\frac{\partial\omega}{\partial y}$$

in the expression for derivative **k** is encountered twice (in the expressions for $\partial k_y/\partial x$ and $\partial k_x/\partial y$), whereas the "compensating" value of the $\partial\omega/\partial t$ derivative is encountered only once in the expression for ω. Thus, the $\partial k_y/\partial x = \partial k_x/\partial y$ condition characterizing the standard field model leads to the systematic error when the exact solution is replaced by the model solution.

We can eliminate the linear discrepancy term by introducing the field model with transverse amplitude modulation; for this purpose, we multiply (4.23) into a certain, still unknown, smooth function $A(y)$:

$$U = A_1(y)\left[A_0 + \frac{\partial A}{\partial x}x + \frac{\partial A}{\partial t}t\right]\exp\left\{i\left(k_x x + \frac{\partial k_y}{\partial x}xy + \frac{\partial k_x}{\partial y}yx - \right.\right.$$
$$\left.\left. - \omega t - \frac{\partial\omega}{\partial y}yt\right)\right\}. \qquad (4.23a)$$

The remaining parameters of the phase and amplitude function in (4.23a) remain the same, as is observed in the standard field model (4.23).

Substituting (4.23a) into Eq. (1.7), we obtain the condition of disappearance of the linear discrepancy term:

$$\frac{\partial^2 A_1}{\partial y^2} - 2\frac{\omega}{c^2}\frac{\partial\omega}{\partial y}A_1 y = 0.$$

The Airy function is the solution to this equation [24]. Note that this equation does not violate the conditions of STRO applicability (4.1) because the scale of y is here within the Fresnel zone $\sim\lambda$, which is substantial when the field in the vicinity of point X_0, Y_0, T_0 is described within the scope of an RO approximation. The Airy function is plotted in Fig. 16.

We have obtained the well-known classical result of exact solution of the wave equation for wave refraction in the layer with linearly changing permittivity [69]. Thereby, we added the second derivative of amplitude $\nabla_\perp^2 A$ to the eikonal equation, concerning the wave equation (1.7a), and introduced a weak transverse field inhomogeneity in the form of the Airy function, concerning the ray series (4.2) we took into account

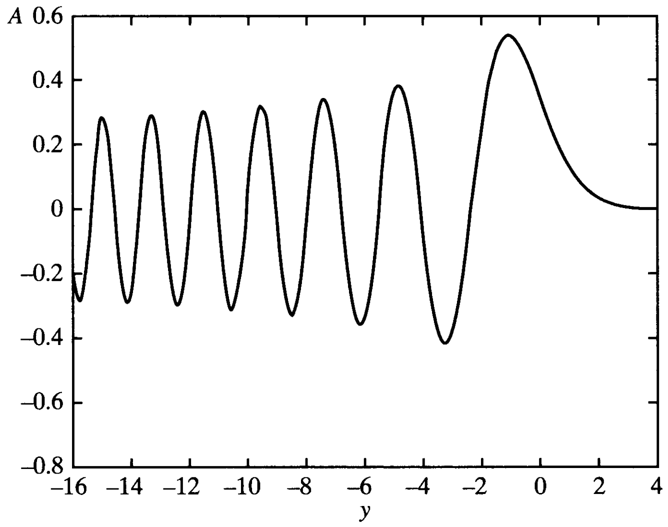

Fig. 16. Airy function as a solution to the equation $\frac{d^2A}{dy^2} - Ay = 0$

the integral effect of the infinite sum of high-order amplitude corrections, which has the scale of refractive effects.

Concerning the standard field model (4.23), the condition

$$\frac{\partial^2 A_1}{\partial y^2} - 2\frac{\omega}{c^2}\frac{\partial \omega}{\partial y}A_1 y = 0$$

is equivalent to simple elimination of the $(dk_y/dx)xy$ term from the phase function, i.e., to introducing the curl component into k:

$$U = \left(A_0 + \frac{\partial A}{\partial x}x + \frac{\partial A}{\partial t}t\right)\exp\left\{i\left(k_x x + \frac{\partial k_x}{\partial y}yx - \omega t - \frac{\partial \omega}{\partial y}yt\right)\right\}. \qquad (4.25)$$

A new model can already be used to describe a wave with transverse frequency modulation because the systematic error in the form of the linear refraction term disappeared from the discrepancy equation (4.24).

For the modified model, $(\mathbf{k}\cdot\nabla)\mathbf{k} = 0$, and the behavior of the group velocity vector in a homogeneous medium is described by the vector total derivative

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2\omega_L^2}{\omega^3}\nabla_\perp \omega, \qquad (4.26)$$

which follows from (4.21).

Here, $\nabla_\perp \omega$ is the index of transverse frequency modulation.

Expression (4.26) describes the effect of dispersion refraction of a wave with transverse frequency modulation in a homogeneous medium. This effect, related to a change in the carrier frequency along the wave front, depends on modulation index $\nabla_{\perp}\omega$ and dispersive properties of the medium. In the absence of dispersion ($\omega_L = 0$), the effect is absent regardless of a wave modulation degree.

We now demonstrate that the scale of the dispersion refraction effect (4.26) corresponds to that of the ordinary refraction effect (4.22) in standard STRO.

Assume that the spatial scales of changes in the medium parameters ω_L and frequency ω are equal to each other:

$$\frac{|\nabla_{\perp}\omega_L|}{\omega_L} = \frac{|\nabla_{\perp}\omega|}{\omega} = \frac{1}{L_P} = \frac{1}{L_W} = \frac{1}{L}.$$

Substituting these derivatives into (4.22) and (4.26), we obtain identical values of the group velocity vector derivative

$$\frac{d|\mathbf{V}_{g\perp}|}{dt} = \frac{c^2\omega_L^2}{\omega^2 L}.$$

We now consider a more general situation, when a wave with transverse frequency modulation propagates in an arbitrary stationary inhomogeneous medium $\omega_L(\mathbf{r})$. We begin with the standard field model.

We specify the following parameters of the wave field and medium at point X_0, Y_0, T_0: carrier frequency ω, frequency gradient $\nabla\omega = \mathbf{e}_y(\partial\omega/\partial y)$, medium eigenfrequency ω_L, and medium gradient $\nabla\omega_L = \mathbf{e}_x(\partial\omega_L/\partial x) + \mathbf{e}_y(\partial\omega_L/\partial y)$. Let the $\mathbf{k}$ vector be directed along the x axis.

In accordance with (4.11), vector $\mathbf{k}$ derivatives have the following components:

$$\frac{\partial\mathbf{k}}{\partial x} = \mathbf{e}_x\left(-\frac{\omega_L}{c^2k_x}\frac{\partial\omega_L}{\partial x}\right) + \mathbf{e}_y\left(\frac{\omega}{c^2k_x}\frac{\partial\omega}{\partial y} - \frac{\omega_L}{c^2k_x}\frac{\partial\omega_L}{\partial y}\right).$$

From the $\nabla\times\mathbf{k} = 0$ condition it follows that

$$\frac{\partial\mathbf{k}}{\partial y} = \mathbf{e}_x\left(\frac{\omega}{c^2k_x}\frac{\partial\omega}{\partial y} - \frac{\omega_L}{c^2k_x}\frac{\partial\omega_L}{\partial y}\right) + \mathbf{e}_y\frac{\partial k_y}{\partial y}.$$

The $\partial k_y/\partial y$ derivative defines here a ray divergence. From (4.12) we obtain:

$$\frac{\partial\omega}{\partial t} = 0.$$

The phase function Ψ of the wave field at point $X_0 + x$, $Y_0 + y$, $T_0 + t$ can be written as

$$\Psi = \Psi_0 + k_x x + \frac{1}{2}\frac{\partial k_x}{\partial x}x^2 + \frac{\partial k_y}{\partial x}xy + \frac{\partial k_x}{\partial y}yx + \frac{1}{2}\frac{\partial k_y}{\partial y}y^2 -$$
$$- \omega t - \frac{\partial \omega}{\partial y}yt. \tag{4.27}$$

The complete wave function has the following form:

$$U = \left(A_0 + \frac{\partial A}{\partial x}x + \frac{\partial A}{\partial t}t\right)\exp(i\Psi). \tag{4.28}$$

By analogy with the previous example, we substitute Ansatz (4.28) into the wave equation (1.7). An analysis of the obtained discrepancy equation shows that maximum discrepancy between the exact and approximate solutions is caused by the linear term

$$-2\left(\frac{\omega}{c^2}\frac{\partial \omega}{\partial y} - \frac{\omega_L}{c^2}\frac{\partial \omega_L}{\partial y}\right)Ay.$$

By analogy with a homogeneous medium, we can eliminate this systematic error by introducing a smoothly inhomogeneous field model, the amplitude function of which is described with the Airy equation:

$$\frac{\partial^2 A_1}{\partial y^2} - 2\left(\frac{\omega}{c^2}\frac{\partial \omega}{\partial y} - \frac{\omega_L}{c^2}\frac{\partial \omega_L}{\partial y}\right)A_1 y = 0.$$

As in the case of a homogeneous medium, for the phase function this is equivalent to the elimination of the $\partial k_y/\partial x$ derivative from (4.27), as a result of which we obtain the modified phase function Ψ_1:

$$\Psi_1 = \Psi_0 + k_x x + \frac{1}{2}\frac{\partial k_x}{\partial x}x^2 + \frac{\partial k_x}{\partial y}yx + \frac{1}{2}\frac{\partial k_y}{\partial y}y^2 - \omega t - \frac{\partial \omega}{\partial y}yt. \tag{4.29}$$

For the modified field model with the phase function (4.29), the wavevector derivative (4.21), characterizing refraction effects, is expressed as

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2\omega_L}{\omega^2}\nabla\omega_L - \frac{c^2\omega_L^2}{\omega^3}\nabla_\perp\omega + \frac{c^2\omega_L^3}{\omega^4}\nabla_\perp\omega_L. \tag{4.30}$$

For this model (as well as for the standard model), from (4.9) and (4.29) it follows that

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \mathbf{V}_g \cdot \nabla\omega = 0.$$

Two additional components appeared in the equation for the group velocity vector derivative in a smoothly inhomogeneous field model (4.30) as compared to the standard model (4.13).

The expression

$$-\frac{c^2\omega_L^2}{\omega^3}\nabla_\perp\omega$$

describes the dispersion refraction effect, and the expression

$$+\frac{c^2\omega_L^3}{\omega^4}\nabla_\perp\omega_L$$

corrects the value of ordinary refraction for modulated and monochromatic waves.

If a wave is not quasi-monochromatic, in the initial data it is necessary to specify the transverse frequency modulation index $\nabla_\perp\omega$ in addition to the $\mathbf{k}$ and ω parameters, which are used for standard STRO, in order to determine the space-time ray trajectory within the scope of the modified model.

Further transformation of frequency modulation during the propagation process can be calculated from the condition of energy balance within a ray tube, as is performed when amplitude is calculated within the scope of ordinary RO. This method will be considered below, but we should note that the other, more general, method for calculating amplitude and modulation index will be elaborated in the next chapter. Hereafter, we use the $\Omega = -\nabla_\perp\omega$ designation in order to simplify records.

To obtain the variation in the amplitude δA and modulation index $\delta\Omega$ along a ray, we will use the fact that longitudinal dispersion compression (or tension) of a wave packet is not observed in the absence of longitudinal frequency modulation, and the energy flux (average during the period) remains constant in each ray tube section; i.e.,

$$\int_{S_0}\langle P_0\rangle dS_0 = \int_{S_1}\langle P_1\rangle dS_1.$$

In our case the energy flux density $\langle P\rangle$ (4.18) in the $\mathbf{R}_0$ section depends on the traverse coordinate η as:

$$\langle P_0\rangle = \frac{c^2}{2\tau}A_0^2(\omega-\Omega_0\eta)\left(k-\frac{\omega}{c^2k}\Omega_0\eta\right).$$

For the adjacent cross section $\mathbf{R}_1 = \mathbf{R}_0 + \int_{T_0}^{T_1}\mathbf{V}_g dt$, we can write

$$\langle P_1\rangle = \frac{c^2}{2\tau}(A_0+\delta A)^2(\omega-(\Omega_0+\delta\Omega)\eta)\left(k+\delta k-\frac{\omega}{c^2k}(\Omega_0+\delta\Omega)\eta\right).$$

In the two-dimensional case, the energy balance along a ray has the form

$$\int_{-Y_0/2}^{Y_0/2} A_0^2(\omega - \Omega_0\eta)\left(k - \frac{\omega}{c^2 k}\Omega_0\eta\right)d\eta =$$

$$= \int_{-Y_1/2}^{Y_1/2} (A_0 + \delta A)^2(\omega - (\Omega_0 + \delta\Omega)\eta) \times$$

$$\times\left(k + \delta k - \frac{\omega}{c^2 k}(\Omega_0 + \delta\Omega)\eta\right)d\eta. \tag{4.31}$$

Here Y_0 and Y_1 are the ray tube widths in the $\mathbf{R}_0$ and $\mathbf{R}_1$ sections, respectively. From (4.31) we obtain

$$\delta\Omega = \Omega_0 \frac{Y_0 - Y_1}{Y_1},$$

$$\delta A = A_0 \frac{\sqrt{kY_0} - \sqrt{(k + \delta k)\,Y_1}}{\sqrt{(k + \delta k)\,Y_1}}.$$

We now write all ray equations for the smoothly inhomogeneous field model:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_g,$$

$$\mathbf{V}_g = c^2\frac{\mathbf{k}}{\omega},$$

$$\frac{d\omega}{dt} = 0, \tag{4.32}$$

$$\frac{d\mathbf{V}_g}{dt} = -\frac{\omega_L}{\omega^2}\nabla\omega_L - \frac{\omega_L^2}{\omega^3}\nabla_\perp\omega + \frac{\omega_L^3}{\omega^4}\nabla_\perp\omega_L,$$

$$\delta\Omega = \Omega_0 \frac{Y_0 - Y_1}{Y_1},$$

$$\delta A = A_0 \frac{\sqrt{kS_0} - \sqrt{(k + \delta k)S_1}}{\sqrt{(k + \delta k)S_1}}.$$

Here Y_0 and Y_1 are the widths of ray tube cross-sections along the frequency gradient at points $\mathbf{R}_0$, T_0 and $\mathbf{R}_1$, T_1; and S_0, S_1 are the section areas at the same points for the three-dimensional case.

§ 6. Discussion of results

In the fifth paragraph of this chapter, having directly substituted the locally-plane homogeneous monochromatic wave model into the initial wave equation, we confirmed the conclusion that such a model cannot describe all causes of refraction effects, drawn in the fourth paragraph of the first chapter based on the exact solution. Thereby, we used the discrepancy method as a criterion of applicability of the HF asymptotic form.

Plane wave as a field model originates naturally, if only the eikonal and transport equations from the infinite ray series (4.2) are considered. But the conditions of RO applicability (4.1) do not result in that we can consider only two terms of the series when describing refraction in a dispersive medium. At least, it is necessary to determine the value to which the rejected terms converge.

For example, in the particular case of refraction in the layer with linearly changing permittivity, the exact solution to the wave equation is the wave with a smooth transverse amplitude inhomogeneity in the form of the Airy function, to which the infinite ray series converges. To understand that homogeneous and inhomogeneous waves behave differently from the viewpoint of refractive effects, it is appropriate to recall that any (except linear) transverse inhomogeneity of amplitude changes the wave phase velocity (see Chapter 1).

When we derived the equations of modified STRO, we did not sum up the ray series (which would lead us to the same result) but directly specified the field model in the form of the Airy function, having indicated that the condition

$$\nabla_{\perp}^{2} A - 2\left(\frac{\omega}{c^{2}} \nabla_{\perp} \omega - \frac{\omega_{L}}{c^{2}} \nabla_{\perp} \omega_{L}\right) A y = 0 \tag{4.33}$$

eliminates the linear term of the discrepancy equation for standard STRO, which corresponds to the systematic error with the refractive spatial-temporal scales.

An undoubtedly mathematically correct method for deriving ray equations by expanding the wave field into the infinite series (4.2) cannot be considered successful from the viewpoint of physics since this method masks a number of processes proceeding during wave propagation in dispersive media.

The correction of the model made it possible to obtain the explicit description of dispersion refraction within the scope of RO and to specify the description of ordinary refraction, which will be considered in more detail in Chapter 7 of this monograph.

Concerning Eq. (1.7a), we included the second derivative of the amplitude $\nabla_{\perp}^{2} A$ in the eikonal equation. A certain analogy with the parabolic equation of the diffraction theory is traced here. In the case of the parabolic equation, part of the phase function is included in the complex amplitude function, defining wave direction, and the second derivative of the amplitude function is included in the phase function and also corrects wave energy propagation direction.

Although equation (4.33) specifies the field model, we do not include this equation in the general list of ray equations since we will never need it in the explicit form. However, the "footprint" of a smooth transverse amplitude inhomogeneity is present in the expression for the $d\mathbf{V}_g/dt$ derivative (4.30). If the first term with operator ∇ corresponds to the standard (plane) version of the space-time ray method, the last two terms with operator $\nabla_\perp$ implicitly reflect the degree of transverse inhomogeneity of the wave amplitude considered in the fourth paragraph of Chapter 1.

Certainly, the usage of the Airy function as a wave model does not contradict the conditions of applicability of the ray asymptotic form. It would be strange if the exact solution of the simplest refractive problem contradicted the asymptotic form, which should, in essence, describe refraction effects.

We can considerably simplify the derivation of the equations for the modified STRO version (4.32) by selecting the field model and calculating the group velocity vector and its derivative along a ray. Such a procedure is described in [111]. However, the presence of formula (4.21), which makes it possible to analyze the general form of wave field structures that can result in refraction, should be considered as an advantage of the discussed method.