Chapter 1

Initial wave equations and some of their exact solutions

§ 1. Wave equations

The wave equation for the scalar field function $U(\mathbf{r}, t)$ in an inhomogeneous medium with time dispersion has the following general form [2, 16, 40, 46]:

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\{L(U)\} = 0. \tag{1.1}$$

Here the linear integral operator $L(U)$ is defined as

$$L(U) = \int_{-\infty}^{t} \tilde{\varepsilon}(\mathbf{r}, t-\tau)\, U(\mathbf{r}, \tau)\, d\tau. \tag{1.2}$$

Equation (1.2) corresponds to the dispersion law for a medium where the wave field depends on its values at previous instants $\tau \le t$. For $(t-\tau) < 0$ the kernel $\tilde{\varepsilon}$ (r, $t-\tau$) vanishes owing to the causality principle.

If $U(\mathbf{r}, t)$ is the solution to Eq. (1.1), each spectral field component

$$u(\mathbf{r}, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\mathbf{r}, t)\exp(i\omega t)\, dt \tag{1.3}$$

satisfies the Helmholtz equation

$$\nabla^2 u + \frac{\omega^2}{c^2}\varepsilon u = 0, \tag{1.4}$$

where the $\varepsilon(\mathbf{r}, \omega)$ is the Fourier transform of kernel $\tilde{\varepsilon}(\mathbf{r}, \omega)$:

$$\varepsilon(\mathbf{r}, \omega) = \frac{1}{2\pi}\int_{0}^{\infty} \tilde{\varepsilon}(\mathbf{r}, t)\exp(i\omega t)\, dt. \tag{1.5}$$

Here the integration lower limit is zero because the $\tilde{\varepsilon}$ function becomes zero at $t < 0$.

The $U(\mathbf{r}, t)$ and $u(\mathbf{r}, \omega)$ functions are connected by the inverse Fourier transform

$$U(\mathbf{r},t) = \int_{-\infty}^{\infty} u(\mathbf{r},\omega)\exp(-i\omega t)\,d\omega.$$

In a medium without time dispersion, which corresponds to the singular kernel

$$\tilde{\varepsilon} = \varepsilon_0(\mathbf{r})\delta(t-\tau)$$

wave equation (1.1) is transformed into the d'Alembert equation

$$\nabla^2 U - \frac{\varepsilon_0}{c^2}\frac{\partial^2 U}{\partial t^2} = 0. \tag{1.6}$$

In this case the $\varepsilon(\mathbf{r}, \omega) = \varepsilon_0(\mathbf{r})$ function in Helmholtz equation (1.4) is independent of frequency.

The other special case of generalized equation (1.1) is the Klein–Gordon equation (KGE),

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \frac{\omega_L^2}{c^2}U = 0 \tag{1.7}$$

with the kernel $\tilde{\varepsilon}$ from (1.2)

$$\tilde{\varepsilon}(\mathbf{r},t) = \delta(t) - \frac{\omega_L(\mathbf{r})}{t}J_1[\omega_L(\mathbf{r})t], \tag{1.8}$$

frequency-dependent dielectric permittivity ε from (1.5)

$$\varepsilon(r,\omega) = 1 - \frac{\omega_L^2(\mathbf{r})}{\omega^2} \tag{1.9}$$

and the dispersion relation

$$\omega^2 = \omega_L^2 + k^2c^2. \tag{1.10}$$

Here ω is the circular frequency of a plane homogeneous monochromatic wave, ω_L is the intrinsic angular frequency of the medium oscillators, k is the wavenumber, and c is the wave velocity in free space (for $\omega_L = 0$). The velocity c is a constant for a specific wave motion type independent of a medium. For electromagnetic waves, this is the velocity of light in vacuum. The propagation medium properties are specified via the spatial distribution of parameter ω_L.

We now consider this equation in more detail because the monograph is mostly related to its solutions.

In the literature Eq. (1.7) is also called the Klein–Gordon–Fock equation and the Gordon linearized sine-equation. KGE describes a wide class of wave processes (from mechanical to quantum-mechanical wave processes). The common property of all various wave processes described by

KGE consists in that a "backmoving force", proportional to a disturbing force of opposite sign, exists in a medium. In particular, KGE is used to describe small mechanical oscillations of the system of coupled pendulums (large oscillations are described by the Gordon sine-equation). The smallness criterion consists in that $\sin U$ can be replaced by U. In this case ω_L is the intrinsic frequency of pendulums [2, 9]. Equation (1.7) describes electromagnetic wave propagation in metal waveguides; parameter ω_L corresponds here to the waveguide cutoff frequency. In quantum mechanics KGE is used to describe a scalar (pseudo-scalar) field corresponding to spinless particles (e.g., pseudo-scalar π-mesons). Moreover, KGE is an asymptotic equation for all types of electromagnetic wave dispersion at increasing frequency ω. This becomes clear if we take into account that, at increased frequencies, a wave interacts only with the least inertial charge carriers (electrons) that form an "elastic electron gas" (e.g., in the X-ray range of electromagnetic waves) [13, 14].

The Klein–Gordon equation is probably most popular in the radiophysics field related to propagation of HF (decameter) radiowaves (3–30 MHz) in the Earth's ionosphere. In this case the scalar function U is considered as a component of the electric field vector $\mathbf{E}$, and ω_L has a sense of the Langmuir frequency of ionospheric plasma electrons. Sometimes ω_L is called plasma cutoff frequency. Equation (1.7) describes propagation of TEM-waves without taking magnetized plasma (the geomagnetic field) into account at frequencies when ion motion can be neglected.

In plasma physics such waves are called Langmuir waves. "These waves originate when plasma quasi-neutrality is disturbed; i.e., when electrons are shifted relative to ions. The electric field that originates in this case generates a quasi-elastic force, which tends to return electrons to the state of equilibrium. Since the weight of electrons is much smaller than that of ions, electrons oscillate under the action of this quasi-elastic force when ions are almost motionless" [7].

Let us represent the kernel $\tilde{\varepsilon}$ of the integral operator (1.2) in the following form:

$$\tilde{\varepsilon}(\mathbf{r},t) = \varepsilon_0(\mathbf{r})\delta(t) + h(\mathbf{r},t). \tag{1.11}$$

Such a representation will make it possible to distinguish purely wave effects associated with the δ-function, and purely dispersive effects related to the h function responsible for time dispersion.

We now rewrite wave equation (1.1), taking into account representation (1.11):

$$\nabla^2 U - \frac{\varepsilon_0}{c^2}\frac{\partial^2 U}{\partial t^2} - \frac{1}{c^2}M(U) = 0, \tag{1.1a}$$

where

$$M(U) = \int_{-\infty}^{t} \frac{\partial^2 h}{\partial t^2}(t-\tau)\,U(\tau)\,d\tau = \int_{0}^{\infty} \frac{\partial^2 h}{\partial t^2}(\tau)\,U(t-\tau)\,d\tau. \tag{1.2a}$$

For wave equation (1.1a)–(1.2a), we now obtain the energy balance equation, using the technique described in [9, 11]. For this purpose, we multiply (1.1a)–(1.2a) into derivative $\partial U/\partial t$ and write

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\left(\frac{\partial U}{\partial t}\right)^2 + \int_{-\infty}^{t} \frac{\partial U}{\partial t}(\tau)\,d\tau \int_{-\infty}^{\tau} \frac{\partial^2 h}{\partial t^2}(\tau-\sigma)\,U(\sigma)\,d\sigma\right) -$$

$$-c^2 \frac{\partial U}{\partial t}\,\nabla^2 U = 0. \tag{1.12}$$

Then, we will add the term

$$c^2 \nabla U \frac{\partial(\nabla U)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{c^2}{2}(\nabla U)^2\right)$$

to the left-hand side of (1.12) and will subtract this term. It is clear that

$$-c^2\frac{\partial U}{\partial t}\nabla^2 U - c^2 \nabla U \frac{\partial(\nabla U)}{\partial t} = -c^2\nabla\left(\frac{\partial U}{\partial t}\nabla U\right).$$

Consequently, the energy balance equation for (1.1a)–(1.2a) has the following form:

$$\frac{\partial W}{\partial t} = -\nabla \cdot \mathbf{P},$$

where

$$W = \frac{\varepsilon_0}{2}\left(\frac{\partial U}{\partial t}\right)^2 + \frac{c^2}{2}(\nabla U)^2 +$$

$$+\int_{-\infty}^{t} \frac{\partial U}{\partial t}(\tau)\,d\tau \int_{-\infty}^{\tau} \frac{\partial^2 h}{\partial t^2}(\tau-\sigma)\,U(\sigma)\,d\sigma \tag{1.13}$$

is the wave field energy density, and

$$\mathbf{P} = -c^2 \frac{\partial U}{\partial t}\nabla U \tag{1.14}$$

is the energy flux density.

We obtain the energy balance equation for KGE (1.7) in a similar way, i.e., by multiplying (1.7) into the derivative dU/dt, and rewrite this equation in the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial U}{\partial t}\right)^2 + \frac{1}{2}\omega_L^2 L^2\right) - c^2\frac{\partial U}{\partial t}\,\nabla^2 U = 0. \tag{1.12a}$$

After the above procedures, performed for the general case of arbitrary dispersion, it becomes clear that the energy flux density is defined by the same formula (1.14), and the energy density formula has a simpler form:

$$W = \frac{1}{2}\left\{\left(\frac{\partial U}{\partial t}\right)^2 + c^2(\nabla U)^2 + \omega_L^2 U^2\right\}. \qquad (1.15)$$

In the following paragraphs, we will consider several KGE exact solutions, which are very interesting from the viewpoint of physics of wave processes in dispersive media. These examples will make it possible to understand the monograph materials presented in the other chapters.

§ 2. Undistorted signal transmission in a dispersive medium

An exact solution in the form of an inhomogeneous plane wave exists for Eq. (1.7)

$$U = A_0 \exp(-py) \exp\{i(\omega t - kx)\}, \qquad (1.16)$$

for which the attenuation parameter p, angular frequency ω, and wavenumber k are connected by the dispersion relation for inhomogeneous waves:

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} + p^2. \qquad (1.17)$$

In a special case when $p = \omega_L/c$ (i.e., when the attenuation parameter p is matched with the parameter ω_L of a homogeneous medium), Eq. (1.17) coincides with the dispersion equation for homogeneous waves in free space:

$$k^2 = \frac{\omega^2}{c^2}.$$

Thus, the wave function, which can be represented as a superposition of inhomogeneous waves (1.16) with $p = \omega_L/c$, propagates without dispersive distortions in a medium with time dispersion described by KGE. Indeed, assume that

$$U = F(ct \pm x) \exp\{-py\}, \qquad (1.18)$$

where F is an arbitrary function of argument $(ct \pm x)$, and $p = \omega_L/c$. An identical equality is obtained by substituting (1.18) into (1.16). Note, that the phenomenon of "cutoff" of the LF spectral components with $\omega < \omega_L$, typical of homogeneous waves, is not observed for inhomogeneous waves.

For certain types of waves with dependence of attenuation along the transverse coordinate y more complex than (1.16), it is possible to find an inhomogeneous medium where the wave function, which can be represented as a decomposition into corresponding monochromatic inhomogeneous waves, propagates in a dispersive medium without distortions.

For example, if

$$U = F(ct \pm x)\exp\{\beta y^2\}, \tag{1.19}$$

the ω_L profile matched with the wave has the form

$$\omega_L^2 = c^2(2\beta + 4\beta^2 y^2).$$

If the wave function has the form

$$U = F(ct \pm x)\exp\{py + \gamma y^3\}, \tag{1.20}$$

the matched profile is defined by the expression

$$\omega_L^2 = c^2(p^2 + 6\gamma y + 6p\gamma y^2 + 9\gamma^2 y^4).$$

However, it is easy to show that smooth (with continuous first derivatives) field functions, bounded along the entire y axis, which could be matched with the ω_L profile, are absent. We now prove this assumption using the rule of contraries. Assume that the $f(y)$ function exists. In such a case, the wave field can be represented as

$$U = F(ct \pm x)f(y). \tag{1.21}$$

Substituting U into KGE, we obtain the coordination criterion

$$\frac{\partial^2 f}{\partial y^2}\frac{1}{f} = \frac{\omega_L^2}{c^2}.$$

The following restrictions are imposed on the $f(y)$ function:

1) $(d^2f/dy^2)/f \geq 0$ since ω_L^2 cannot have negative values;

2) $|f(y)| < c_1$; i.e., the $f(y)$ function should be bounded on the entire y axis, c_1 is an arbitrary positive number.

Since we are not interested in the case when $f(y) = \text{const}$, the function should have at least one extremum.

Let us consider three possible variants:

1. Let $f(y_m)$ be a maximum, $f(y_m) > 0$. In such a case $d^2f/dy^2 < 0$, and condition 1 is not satisfied.

2. Let $f(y_m)$ be a minimum, $f(y_m) < 0$. In such a case $d^2f/dy^2 > 0$, and condition 1 is not satisfied.

3. Let $f(y_m)$ be a maximum, $f(y_m) \leq 0$ or a minimum ($f(y_m) \geq 0$). In such a case, from condition 2 it follows that a bending point, where $\partial^2 f/\partial y^2$ changes its sign, should exist. In this case condition 1 is also not satisfied.

Thus, the assumption that a smooth bounded attenuation function can exist leads to contradiction.

Actually, the wave structure of the (1.18) type can exist as a surface wave; therefore, the property of matched inhomogeneous waves to transmit signals without distortions can be used to generate dispersion-free waveguides that can operate at frequencies $\omega < \omega_L$. The impedance surface in a homogeneous plasma can be an example of such a waveguide for electromagnetic waves.

Assume that the waveguide surface is located in the $y = 0$ plane. We specify the electric field strength $\mathbf{E}$ in the $y > 0$ half-space as

$$E_z = E_0 \exp(-py) \exp\{i(\omega t - kx)\}. \tag{1.22}$$

In this case the magnetic field will have the H_x and H components defined by formulas [22, 23, 35]

$$H_x = -\frac{ip}{\omega\mu_0} E_0 \exp(-py) \exp\{i(\omega t - kx)\}, \tag{1.23}$$

$$H_y = -\frac{1}{\omega\mu_0} E_0 \exp(-py) \exp\{i(\omega t - kx)\}, \tag{1.24}$$

where μ_0 is the absolute magnetic permittivity.

An inhomogeneous wave (1.22)–(1.24) can exist only when surface impedance satisfies the condition

$$Z = \frac{E_z}{H_x} = i\frac{\omega\mu_0}{p} = i\frac{\omega\mu_0 c}{\omega_L}. \tag{1.25}$$

It follows from (1.25) that impedance should be purely reactive.

Note that inhomogeneous waves of a diffractive nature show a time dispersion in a dispersion-free medium ($\omega_L = 0$), and the phase velocity of these waves is always lower than c. In turn, the phase velocity of homogeneous waves in a dispersive medium is always higher than c. Inhomogeneous waves in a dispersive medium can have an arbitrary phase velocity varying from 0 to ∞, including c. In the latter case, the effects of dispersion and diffraction are completely neutralized.

§ 3. Circular stationary waves

If the $\omega = \omega_L$ and $k_\varphi = \pm n$ conditions are satisfied, an inhomogeneous stationary wave

$$U(r, \phi) = r^n \exp\{i(\omega t - k_\varphi \phi)\} \tag{1.26}$$

is defined by Eq. (1.7) written in the cylindrical coordinate system (r, φ, z) [24]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \frac{\omega_L^2}{c^2}U = 0. \tag{1.7a}$$

Formula (1.26) describes a stationary wave, rotating about a center $r = 0$ with an angular wavenumber $k_\varphi = \pm n$, in a homogeneous medium. The case when $k_\varphi = n = 0$ should be excluded because wave motion is absent in this case.

The results of the previous section rather simply explain why an isolated inhomogeneous wave, rotating about a certain center, can exist in a homogeneous medium. Such a wave can evidently exist only when a local phase velocity of this wave is zero at the center and linearly increases along the radius. In this case the local attenuation coefficient p_L should vary from ∞ to 0 if distance changes from 0 to ∞ and $\omega = \omega_L$.

It is easy to make sure that this property is typical of the attenuation coefficient of a locally plane inhomogeneous wave, the form of which can be used to represent (1.26)

$$U(r,\phi) = r^n \exp\{i(\omega t - k_\phi \phi)\} = \exp(p_L r)\exp\{i(\omega t - k_\phi \phi)\},$$

$$p_L = n\frac{\ln r}{r}.$$

Waves of the (1.26) type have a singularity at the origin of coordinates (at $n < 0$) or at infinity (at $n > 0$). Since the considered wave process exists only at $\omega = \omega_L$, this process can be used to determine plasma electron concentration in open resonators at $n < 0$.

As in the previous case, an inhomogeneous wave (1.26) can exist in the form of a surface wave above the impedance boundary. Let us introduce the cylindrical coordinate system and consider the surface of a cylinder, oriented along the z axis as a boundary. Assume that the electric field strength E_z in the $r \geq r_0$ space, where r_0 is a cylinder radius, is specified as

$$E_z = E_0 r^n \exp\{i(\omega t - n\phi)\}.$$

In this case the magnetic field will have the H_r and H_φ components:

$$H_r = \frac{n}{\omega\mu_0} E_0 r^{n-1} \exp\{i(\omega t - n\phi)\},$$

$$H_\phi = -\frac{in}{\omega\mu_0} E_0 r^{n-1} \exp\{i(\omega t - n\phi)\}.$$

As in the previous case, surface impedance has a purely reactive character:

$$Z = i\frac{\omega\mu_0 r_0}{n}.$$

For the waveguide and resonator, the surface structures with the required properties can be represented in the form of a ribbed metal surface or a dielectric layer over a metal substrate [22, 23, 35, 70].

We now show that a rotating isolated wave without singularities can also exist without waveguide surfaces in an inhomogeneous propagation medium. For this purpose, we will seek the solution to Eq. (1.7a) in the form

$$U = A(r)\exp\{i(\omega t - k_\phi \phi)\}. \tag{1.27}$$

We represent an inhomogeneous medium as

$$\omega_L^2 = \omega_{L0}^2\{1 + F(r)\}. \tag{1.28}$$

This medium can be physically realized at

$F(r) \geq -1.$

Substituting (1.27) and (1.28) into (1.7a), we obtain the relationship between the wave and medium parameters:

$$F(r) = \frac{c^2}{\omega_{L0}^2}\left\{\frac{1}{r}\frac{\partial A}{\partial r}A^{-1} + \frac{\partial^2 A}{\partial r^2}A^{-1} - \frac{k_\phi^2}{r^2}\right\}. \tag{1.29}$$

By specifying, e.g., the amplitude $A(r)$ distribution as

$$A(r) = A_0\frac{r}{1+r^2},$$

it is easy to make sure that the wave exists under the following conditions:

$$k_\phi = 1,$$

$$\frac{c^2}{\omega_{L0}^2} = 0.125,$$

$$F(r) = -\frac{1}{(1+r^2)^2}.$$

For plasma, this means that an inhomogeneity should be cylindrically symmetric with a decreased electron concentration at the center.

§ 4. Discussion of results

The KGE exact solutions presented above make it possible to draw the following conclusions:

1. Any transverse field inhomogeneity (except the linear function with a zero second derivative) "blooms" plasma by decreasing the "cut-off" frequency ω_L (1.17). This fact can become of interest when the methods of communication, with a spacecraft descending within a shielding plasma layer, are developed.

2. Any transverse field inhomogeneity (except the linear function) changes the phase velocity $V_p = \omega/k$. These changes can have the scales of diffraction and refraction effects. For example, dispersion effects are compensated by diffraction ones in wave (1.18), which makes it possible to transmit signals in plasma at the velocity of light c without distortions.

In wave (1.19) and (1.20), refraction effects are compensated due to a transverse inhomogeneity of the field amplitude. Indeed, if we, e.g., set $1/\beta \gg \lambda$ in (1.19) (λ is the characteristic wavelength), we will have a smoothly inhomogeneous medium with the spatial scales corresponding to the scales of refraction effects. The standard version of space-time RO, based on a locally plane homogeneous wave model, erroneously demonstrates that refraction is present in this situation, whereas the exact solution indicates that refraction is absent and the phase and group velocities are equal to the velocity of light c.

Examples (1.19) and (1.20) indicate that a standard locally plane homogeneous wave model cannot be used to correctly describe refraction because the error of this model can sometimes be equal to refraction effects. Refraction can be completely described only if at least a certain parameter, responsible for transverse inhomogeneity of the wave field, is present in the field model. This parameter should also enter into the refractive index n.

3. Waves with a stationary energy center and variable propagation directions of all wave zones exist in a homogeneous dispersive medium along with waves propagating from a source to infinity. Exact solution (1.26) makes it possible to state that a wave can reverse its propagation in a dispersive medium. If we cut wave field (1.26) by a plane at $r = 0$ and specify necessary boundary conditions on this plane, a field without singularities will exist in the space without a point $r = 0$. In this case wave energy propagates from one side of the plane to another side in opposite directions.

4. The usage of the rotating wave effect in an open resonator in order to determine the ionospheric plasma electron concentration will make it possible not only to determine the concentration value near a sensor but also to estimate the influence of a sensor on a measured parameter. The measurement scheme can be as follows: a signal with a variable frequency is transmitted from a generator to feeding probes of a resonator. A wave process originates when a generator frequency coincides with ω_L, and a signal appears at receiving probes. By changing circular wave modes n, it is possible to localize field energy at different distances from a sensor, and a change in the resonator excitation frequency will demonstrate the degree of inhomogeneity of the environment.

5. Resonators with a decreased electron concentration at the center of the (1.29) type can originate at ionospheric inhomogeneities stretched along the geomagnetic field.