Chapter 2

Wave packet propagation in a homogeneous half-space with time dispersion

§ 1. Introduction

In this chapter, we will obtain exact expressions for certain integral characteristics of wave packets, which propagate in a homogeneous half-space and are described by KGE.

We will show that certain structures of wave packets lead to a change in the packet propagation direction (to a transverse shift) due to dispersion or refraction effects.

For this purpose, we will solve the problem with boundary conditions in the $x \geq 0$ half-space in the x, y, and t orthogonal coordinates. Here x is the longitudinal coordinate, y is the transverse coordinate, and t is time.

In a general case, the wave field in two- or three-dimensional media is described by the function of three or four variables (two or three spatial coordinates and time). The integral field characteristics related to wave energy motion in the space are introduced in order to simplify the description of the entire wave packet or its parts. The group velocity vector, energy center motion trajectory, and wave packet length and width are usually among these characteristics.

The group velocity vector $\mathbf{V}_g$ in the most general form is introduced as the ratio of the average (during the oscillation period T) energy flux density vector $\mathbf{P}$ to the average (during the same period) energy density W:

$$\mathbf{V}_g = \frac{\int_0^T \mathbf{P} dt}{\int_0^T W dt}. \tag{2.1}$$

The group velocity vector physically specifies the average energy propagation direction and velocity at a certain spatial point.

Such a definition has the sense only when the wave parameters change sufficiently slowly; i.e., when a wave locally behaves as an almost plane and monochromatic wave. In this case the group velocity vector actually describes energy motion of wave packet parts over large distances, which forms the basis for the space-time RO [11, 38, 47].

An alternative version of the wave packet integral description is used in the method of moments, when the characteristics of spatial energy distribution are introduced. The first and second moments (which are usually called the wave packet energy center, length, and width) are used most often [19–21].

The first moments, characterizing the packet energy center, are specified by the following expressions:

$$X_c(t) = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wxdxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wxdxdy}, \tag{2.2a}$$

$$Y_c(t) = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wydxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wydxdy}. \tag{2.2b}$$

The time variations in these moments reflect the average motion of a wave packet.

The second moments, the length σ_x and width σ_y of the wave packet, are defined as

$$\sigma_x^2(t) = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wx^2dxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wdxdy} - X_c^2, \tag{2.3a}$$

$$\sigma_y^2(t) = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wy^2dxdy}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wdxdy} - Y_c^2 \tag{2.3b}$$

and show a change in the packet form.

Both methods cannot be used in the pure form within the scope of the stated problem.

The group velocity vector is unsuitable because we do not intend to impose restrictions on the wave packet parameters, including the condition of quasi-monochromaticity.

The method of moments (2.2a)–(2.3b) cannot be used since the integration over x averages the wave field along the axial coordinate and does not exactly represent the effect of boundary conditions on the packet evolution depending on distance from the boundary.

A rejection of the integration over x in (2.2b) and (2.3b) is a possible modification of the method of moments. As a result, the function of two variables, which describes a wave centered along y depending on x and t, is obtained. However, in this case we do not represent the wave packet as a single whole (i.e., as a quasi-particle) and cannot distinctly characterize the packet evolution depending on distance from the boundary. Only function of one variable x can be used to characterize this evolution.

Therefore, below we modify these two methods in order to avoid the above complexities and solve the problem in the general form.

§ 2. Energy transfer integral characteristics

The average group velocity vector $\mathbf{V}_{ga}(x)$ is defined as

$$\mathbf{V}_{ga} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{P} dy dt}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W dy dt}. \tag{2.4}$$

The $\mathbf{V}_{ga}(x)$ vector is calculated by averaging the wave energy along the y and t axes and consequently characterizes the average direction and velocity of the entire wave packet depending on the axial coordinate x.

To avoid difficulties typical of the method of moments, we will modify formulas (2.2a)–(2.2b) and (2.3a)–(2.3b) by taking an integral with respect to the t variable rather than over the x axis.

We will define the energy center transverse coordinate $Y_{ca}(x)$ (which is hereafter called a "transverse packet coordinate" for simplicity) as follows:

$$Y_{ca} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W y dy dt}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W dy dt}. \tag{2.5}$$

In a similar way, we will define the time of energy center propagation $T_{ca}(x)$ (hereafter, "packet propagation time"):

$$T_{ca} = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wtdydt}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wdydt}, \tag{2.6}$$

and packet width $\sigma_{ya}(x)$:

$$\sigma_{ya}^2 = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wy^2dydt}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wdydt} - Y_{ca}^2, \tag{2.7}$$

and duration $\sigma_{ta}(x)$:

$$\sigma_{ta}^2 = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wy^2dydt}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Wdydt} - T_{ca}^2. \tag{2.8}$$

The integral field characteristics (2.5)–(2.8) adapt the method of moments to the stated Dirichlet problem and make it possible to trace the packet evolution along the x axis. As in a usual method of moments, we consider the packet as a quasi-particle characterized by the transverse coordinate $Y_{ca}(x)$ and propagation time $T_{ca}(x)$. This is a direct analogy with a usual parameterization $X_c(t)$, $Y_c(t)$, but the independent variable t is replaced by x in our case.

A similar analogy is observed between σ_y (2.3b) and σ_{ya} (2.7). Both parameters have the same physical sense: they characterize the wave packet width.

It is to a certain extent difficult to deal with the packet length σ_x (2.3a). This parameter cannot altogether be determined using the selected integration method. On the other hand, we can determine the packet duration σ_{ta} (2.8), which cannot be made using a usual method, and relate the packet length to its period via the average group velocity $\mathbf{V}_{ga}$ defined in (2.4).

Subsequently, we will calculate characteristics (2.4)–(2.8) as functionals of the initial wave field at the $x = 0$ boundary.

§ 3. Problem statement

Assume that a wave packet propagates in the $x \geq 0$ half-plane of the two-dimensional space (x, y). At the $x = 0$ boundary, the wave field is specified as a real-valued function $U(0, y, t)$, which is defined on the entire y and t axes.

We represent the initial function at $x = 0$ using a two-dimensional integral Fourier transform [10, 103]:

$$U(0,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_0(k_y,\omega)\exp\{i(k_y y - \omega t)\}\, dk_y d\omega,$$

where

$$F_0(k_y,\omega) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(0,y,t)\exp\{-i(k_y y - \omega t)\}\, dy dt. \tag{2.9}$$

The $F_0\exp\{i(k_y y - \omega t)\}$ spectral component can be considered as a plane monochromatic wave projection onto the $x = 0$ boundary. Here ω is the angular frequency, k_y is the wave vector projection on the y axis, and the $F_0(k_y, \omega)$ function is the 2D angular-frequency spectrum of the initial field distribution.

Subsequently, we will write spectral operator (2.9) as $\hat{F}\{U\}$.

If we specify the angular-frequency spectrum $F\{x, k_y, \omega\}$ dependent on the longitudinal coordinate x, the $U(x, y, t)$ function will be represented by the expression

$$U(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,k_y,\omega)\exp\{i(k_y y - \omega t)\}\, dk_y d\omega. \tag{2.10}$$

Since the U values for all x should satisfy initial wave equation (1.7), we obtain the equation for the $F\{x, k_y, \omega\}$ spectrum by substituting (2.10) into (1.7):

$$\frac{d^2F}{dx^2} + \left(\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2\right)F = 0. \tag{2.11}$$

The solution to Eq. (2.11), satisfying the $F(0, k_y, \omega) = F(x, k_y, \omega)$ boundary condition and the condition of emission at infinity, has the following form:

$$F = F_0\exp\left(ix\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\right). \tag{2.12}$$

It follows from (2.12) that the angular-frequency spectrum F consists of homogeneous and inhomogeneous waves. If the $K = \omega^2/c^2 - \omega_L^2/c^2 - k_y^2$ value within the square root in the right-hand side of (2.12) is positive, expression (2.12) corresponds to homogeneous plane waves, and $\sqrt{K}$ is the wave vector projection onto the x axis. Negative K values correspond to decaying inhomogeneous waves, and $\sqrt{-K}$ characterizes wave attenuation.

§ 4. Average group velocity vector

In this and next sections, we calculate all introduced energy characteristics, including the average group velocity vector, in the spectral domain. For this purpose, we use the Parseval theorem [103]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_1 U_2 dydt = 4\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1 F_2^* dk_y d\omega. \tag{2.13}$$

Here U_1 and U_2 are the real-valued functions of the y and t variables; F_1 and F_2 are the corresponding complex spectral functions of the k_y and ω variables, and the asterisk * means complex conjugation.

Note that the F spectrum of the real-valued function U has the following properties:

$$F(k_y, \omega) = F^*(-k_y, -\omega),$$

$$F(k_y, -\omega) = F^*(-k_y, \omega).$$

The spectrum of the U wave function is represented by formula (2.12). Using the well-known properties of the Fourier transform [24], it is rather easy to find the spectra of the $\partial U/\partial x$, $\partial U/\partial y$, and $\partial U/\partial t$ derivatives:

$$\hat{F}\left(\frac{\partial U}{\partial x}\right) = i\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\,F, \tag{2.14}$$

$$\hat{F}\left(\frac{\partial U}{\partial y}\right) = ik_y F, \tag{2.15}$$

$$\hat{F}\left(\frac{\partial U}{\partial t}\right) = -i\omega F. \tag{2.16}$$

We now write the integrated components of energy flux vector (1.14) and energy density (1.13) using the spectral field functions:

$$\ll P_x \gg = -c^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\partial U}{\partial x}\frac{\partial U}{\partial t} dydt =$$

$$= 4\pi^2 c^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \omega\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\, FF^* \, dk_y d\omega, \qquad (2.17)$$

$$\ll P_y \gg = -c^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\partial U}{\partial y}\frac{\partial U}{\partial t} dydt =$$

$$= 4\pi^2 c^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} k_y \omega FF^* \, dk_y d\omega, \qquad (2.18)$$

$$\ll W \gg = \frac{1}{2} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left\{ \left(\frac{\partial U}{\partial t}\right)^2 + c^2\left(\frac{\partial U}{\partial x}\right)^2 + c^2\left(\frac{\partial U}{\partial y}\right)^2 + \omega_L^2 U^2 \right\} dydt =$$

$$= 4\pi^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \omega^2 FF^* dk_y d\omega \qquad (2.19)$$

and rewrite (2.17)–(2.19) in terms of the F_0 initial spectrum using expression (2.12) for F. It is convenient to divide the integration limits into two regions ($K \geq 0$ and $K < 0$), corresponding to homogeneous and inhomogeneous waves:

$$\ll P_x \gg = 4\pi^2 c^2 \iint\limits_{K\geq 0} \omega\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\, | F_0 |^2 dk_y d\omega +$$

$$+ 4\pi^2 i c^2 \iint\limits_{K<0} \omega\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\, | F_0 |^2 \times$$

$$\times \exp\left\{ -2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}} \right\} dk_y d\omega, \qquad (2.17a)$$

$$\ll P_y \gg = 4\pi^2 c^2 \iint\limits_{K\geq 0} k_y \omega | F_0 |^2 dk_y d\omega +$$

$$+ 4\pi^2 c^2 \iint\limits_{K<0} k_y \omega | F_0 |^2 \exp\left\{ -2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}} \right\} dk_y d\omega, \qquad (2.18a)$$

$$\ll W \gg = 4\pi^2 \iint\limits_{K\geq 0} \omega^2 | F_0 |^2 dk_y d\omega +$$

$$+ 4\pi^2 \iint\limits_{K<0} \omega^2 | F_0 |^2 \exp\left\{ -2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}} \right\} dk_y d\omega. \qquad (2.19a)$$

It is clear that the imaginary part is present in the expression for $\ll P_x \gg$ (2.17a). However, the value of this part is zero because of the Fourier transform properties for the real function.

We can simplify formulas (2.17a)–(2.19a) by writing the I_i constants for the integrals independent of the x coordinate and the $\alpha_i(x)$ functions for the integrals dependent on x.

In the general case, the components of the average group velocity vector depend on the x longitudinal coordinate and are defined as

$$V_{xga} = \frac{\ll P_x \gg}{\ll W \gg} = \frac{I_1}{I_3 + \alpha_3(x)}, \tag{2.20}$$

$$V_{yga} = \frac{\ll P_y \gg}{\ll W \gg} = \frac{I_2 + \alpha_2(x)}{I_3 + \alpha_3(x)}. \tag{2.21}$$

Here

$$I_1 = 4\pi^2 c^2 \iint\limits_{K \geq 0} \omega \sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2} \, |F_0|^2 dk_y d\omega, \tag{2.20a}$$

$$I_2 = 4\pi^2 c^2 \iint\limits_{K \geq 0} k_y \omega |F_0|^2 dk_y d\omega, \tag{2.21a}$$

$$\alpha_2(x) = 4\pi^2 c^2 \iint\limits_{K<0} k_y \omega |F_0|^2 \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega, \tag{2.21b}$$

$$I_3 = 4\pi^2 \iint\limits_{K \geq 0} \omega^2 |F_0|^2 dk_y d\omega, \tag{2.21c}$$

$$\alpha_3(x) = 4\pi^2 \iint\limits_{K<0} \omega^2 |F_0|^2 \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega. \tag{2.21d}$$

The $\alpha_i(x)$ functions always tend to zero at $x \to \infty$; however, the specific behavior of these functions, characterized by the initial spectrum of decaying waves, can be substantially different, including the possible sign reversal.

§ 5. Wave packet transverse coordinate

To calculate average transverse coordinate (2.5), it is necessary to find the spectral function of the following expression:

$$Wy = \frac{1}{2}\left\{\left(\frac{\partial U}{\partial t}\right)^2 + c^2\left(\frac{\partial U}{\partial x}\right)^2 + c^2\left(\frac{\partial U}{\partial y}\right)^2 + \omega_L^2 U^2\right\}y =$$

$$= \frac{1}{2}\left\{\left(y\frac{\partial U}{\partial t}\right)\left(\frac{\partial U}{\partial t}\right) + c^2\left(y\frac{\partial U}{\partial x}\right)\left(\frac{\partial U}{\partial x}\right) + \right.$$

$$\left. + c^2\left(y\frac{\partial U}{\partial y}\right)\left(\frac{\partial U}{\partial y}\right) + \omega_L^2 (yU)\,U\right\}. \qquad (2.22)$$

For this purpose, we will find the spectra of the $y(\partial U/\partial t)$, $y(\partial U/\partial x)$, $y(\partial U/\partial y)$, and yU functions using the well-known property of the Fourier transform: function multiplication into y is equivalent to function spectrum differentiation with respect to k_y and multiplication into i. When writing the spectrum, we take into account that $F_0 = |F_0| \exp\{i\psi_0\}$:

$$\hat{F}\left\{y\frac{\partial U}{\partial t}\right\} = \omega\frac{\partial |F_0|}{\partial k_y}\exp\{\cdot\} + i\omega|F_0|\frac{\partial \psi_0}{\partial k_y}\exp\{\cdot\} -$$

$$- i|F_0|k_y\omega x\frac{c}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}\exp\{\cdot\}, \qquad (2.23)$$

$$\hat{F}\left\{y\frac{\partial U}{\partial y}\right\} = -|F_0|\exp\{\cdot\} - k_y\frac{\partial |F_0|}{\partial k_y}\exp\{\cdot\} - ik_y|F_0|\frac{\partial \psi_0}{\partial k_y}\exp\{\cdot\} +$$

$$+ ik_y^2|F_0|x\frac{c}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}\exp\{\cdot\}, \qquad (2.24)$$

$$\hat{F}\left\{y\frac{\partial U}{\partial x}\right\} = k_y\frac{c}{\sqrt{\omega - \omega_L^2 - c^2k_y^2}}|F_0|\exp\{\cdot\} -$$

$$- \sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\,\frac{\partial |F_0|}{\partial k_y}\exp\{\cdot\} -$$

$$- i\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\,|F_0|\frac{\partial \psi_0}{\partial k_y}\exp\{\cdot\} + ik_y x|F_0|\exp\{\cdot\}, \qquad (2.25)$$

$$F\{yU\} = i\frac{\partial|F_0|}{\partial k_y}\exp\{\cdot\} - |F_0|\frac{\partial\psi_0}{\partial k_y}\exp\{\cdot\} +$$

$$+|F_0|k_y x\frac{c}{\sqrt{\omega^2-\omega_L^2-c^2k_y^2}}\exp\{\cdot\}. \qquad (2.26)$$

To simplify formulas (2.23)–(2.26) and the next expressions, we introduce the following abbreviation:

$$\{\cdot\} = \left\{i\psi_0 + ix\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\right\}.$$

Using the Parseval theorem (2.13), and expressions (2.23) – (2.26) for the spectrum, we can transform the numerator of formula (2.5) into the following form:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Wydydt = 4\pi^2 \iint_{K\geq 0}\left\{-|F_0|^2\frac{\partial\psi_0}{\partial k_y}\omega^2 +\right.$$

$$\left.+|F_0|^2k_y\omega^2 x\frac{c}{\sqrt{\omega^2-\omega_L^2-c^2k_y^2}}\right\}dk_y d\omega -$$

$$-4\pi^2 \iint_{K<0} |F_0|^2\frac{\partial\psi_0}{\partial k_y}\omega^2\exp\left\{-2x\sqrt{k_y^2+\frac{\omega_L^2}{c^2}-\frac{\omega^2}{c^2}}\right\}dk_y d\omega. \qquad (2.27)$$

Only the real part of the integral is presented in formula (2.27) because the integral imaginary part is zero for real functions.

We now rewrite (2.27) in a more compact form:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Wydydt = I_4 + I_5 x + \alpha_4(x). \qquad (2.27a)$$

Here

$$I_4 = -4\pi^2 \iint_{K\geq 0} |F_0|^2\frac{\partial\psi_0}{\partial k_y}\omega^2 dk_y d\omega, \qquad (2.27b)$$

$$I_5 = 4\pi^2 \iint_{K\geq 0} |F_0|^2 k_y\omega^2\frac{c}{\sqrt{\omega^2-\omega_L^2-c^2k_y^2}}dk_y d\omega, \qquad (2.27c)$$

$$\alpha_4(x) = -4\pi^2 \iint\limits_{K<0} |F_0|^2 \frac{\partial\psi_0}{\partial k_y} \times$$

$$\times\, \omega^2 \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega. \tag{2.27d}$$

The dependence of the wave packet traverse coordinate on x can finally be written as

$$Y_{ca} = \frac{I_4 + I_5 x + \alpha_4(x)}{I_3 + \alpha_3(x)}. \tag{2.28}$$

§ 6. Wave packet propagation time

We now transform the integrand for the numerator in formula (2.6)

$$Wt = \frac{1}{2}\left\{\left(\frac{\partial U}{\partial t}\right)^2 + c^2\left(\frac{\partial U}{\partial x}\right)^2 + c^2\left(\frac{\partial U}{\partial y}\right)^2 + \omega_L^2 U^2\right\} t$$

into the following form:

$$Wt = \frac{1}{2}\left\{\left(t\frac{\partial U}{\partial t}\right)\left(\frac{\partial U}{\partial t}\right) + c^2\left(t\frac{\partial U}{\partial x}\right)\left(\frac{\partial U}{\partial x}\right) +\right.$$

$$\left. + c^2\left(t\frac{\partial U}{\partial y}\right)\left(\frac{\partial U}{\partial y}\right) + \omega_L^2 (tU)\, U\right\}. \tag{2.29}$$

Using the fact that multiplication of the function into t in the space-time region is equivalent to differentiation of the function spectrum with respect to ω and to multiplication of this spectrum into $-i$, we can calculate the spectra of the $t(\partial U/\partial t)$, $t(\partial U/\partial x)$, $t(\partial U/\partial y)$, and tU functions.

From (2.14)–(2.16) we obtain that

$$\hat{F}\left\{t\frac{\partial U}{\partial t}\right\} = -|F_0|\exp\{\cdot\} - \omega\frac{\partial|F_0|}{\partial\omega}\exp\{\cdot\} - i\omega|F_0|\frac{\partial\psi_0}{\partial\omega}\exp\{\cdot\} -$$

$$- i\frac{\omega^2}{c}|F_0|\, x\frac{1}{\sqrt{\omega^2 - \omega_L^2 - c^2 k_y^2}}\exp\{\cdot\}; \tag{2.30}$$

$$\hat{F}\left\{t\frac{\partial U}{\partial y}\right\} = k_y\frac{\partial|F_0|}{\partial\omega}\exp\{\cdot\} + ik_y|F_0|\frac{\partial\psi_0}{\partial\omega}\exp\{\cdot\} +$$

$$+ ik_y\frac{\omega^2}{c}|F_0|\, x\frac{1}{\sqrt{\omega^2 - \omega_L^2 - c^2 k_y^2}}\exp\{\cdot\}; \tag{2.31}$$

$$\hat{F}\left\{t\frac{\partial U}{\partial x}\right\} = \frac{\omega}{c}|F_0|\frac{1}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}\exp\{\cdot\} +$$

$$+\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}\frac{\partial|F_0|}{\partial\omega}\exp\{\cdot\} +$$

$$+i\sqrt{\frac{\omega^2}{c^2} - \frac{\omega_L^2}{c^2} - k_y^2}|F_0|\frac{\partial\psi_0}{\partial\omega}\exp\{\cdot\} + i\frac{\omega}{c^2}|F_0|x\exp\{\cdot\}; \qquad (2.32)$$

$$\hat{F}\{tU\} = -i\frac{\partial|F_0|}{\partial\omega}\exp\{\cdot\} + |F_0|\frac{\partial\psi_0}{\partial\omega}\exp\{\cdot\} +$$

$$+\frac{\omega}{c}|F_0|x\frac{1}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}\exp\{\cdot\}. \qquad (2.33)$$

Using the Parseval theorem with expressions (2.30)–(2.33) for the spectra, we transform the numerator of formula (2.6) into the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Wtdydt = 4\pi^2 \iint_{K\geq 0} \left\{\omega^2|F_0|^2\frac{\partial\psi_0}{\partial\omega} + \right.$$

$$\left. + \frac{1}{c}|F_0|^2 x\frac{\omega^3}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}\right\} dk_y d\omega +$$

$$+ 4\pi^2 \iint_{K<0} \omega^2|F_0|^2\frac{\partial\psi_0}{\partial\omega}\exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\}dk_y d\omega \qquad (2.34)$$

which makes it possible to write the packet propagation time as

$$T_{ca} = \frac{I_6 + I_7x + \alpha_5(x)}{I_3 + \alpha_3(x)}. \qquad (2.35)$$

Here, we introduce the following abbreviations:

$$I_6 = 4\pi^2 \iint_{K\geq 0} \omega^2|F_0|^2\frac{\partial\psi_0}{\partial\omega}dk_y d\omega; \qquad (2.35a)$$

$$I_7 = \frac{4\pi^2}{c} \iint_{K\geq 0} |F_0|^2\frac{\omega^3}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}}dk_y d\omega; \qquad (2.35b)$$

$$\alpha_5(x) = 4\pi^2 \iint\limits_{K<0} \omega^2 |F_0|^2 \frac{\partial\psi_0}{\partial\omega} \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega. \qquad (2.35c)$$

In the general case, the wave packet propagation time $T_{ca}(x)$ (2.35) and traverse coordinate Y_{ca} (2.28) nonlinearly depend on distance x since the packet of decaying waves exists in the spectrum.

§ 7. Wave packet width and duration

Using the procedures similar to those described above, we can obtain the expressions for the wave packet width σ_{ya} (2.7) and duration σ_{ta} (2.8):

$$\sigma_{ya}^2 = \frac{I_8 + I_9 x + I_{10} x^2 + \alpha_6(x) + x\alpha_7(x) + x^2\alpha_8(x)}{I_3 + \alpha_3(x)} - Y_{ca}^2; \qquad (2.36)$$

$$\sigma_{ta}^2 = \frac{I_{11} + I_{12} x + I_{13} x^2 + \alpha_9(x) + x\alpha_{10}(x) + x^2\alpha_{11}(x)}{I_3 + \alpha_3(x)} - T_{ca}^2. \qquad (2.37)$$

Here, the I_i constants and $\alpha_i(x)$ functions in (2.36) are defined by the initial condition spectrum on the $x = 0$ boundary in the following way:

$$I_8 = 4\pi^2 \iint\limits_{K\geq 0} \left\{\omega^2\left(\frac{\partial|F_0|}{\partial k_y}\right)^2 + \omega^2|F_0|^2\left(\frac{\partial\psi_0}{\partial k_y}\right)^2 + \right.$$

$$\left. + \frac{1}{2}c^2|F_0|^2\left(\frac{c^2k_y^2}{\omega^2 - \omega_L^2 - c^2k_y^2} + 1\right)\right\} dk_y d\omega; \qquad (2.36a)$$

$$I_9 = -8\pi^2 \iint\limits_{K\geq 0} k_y\omega^2|F_0|^2 \frac{\partial\psi_0}{\partial k_y} \frac{c}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}} dk_y d\omega; \qquad (2.36b)$$

$$I_{10} = 2\pi^2 c^2 \iint\limits_{K\geq 0} k_y^2|F_0|^2\left(\frac{\omega^2 + \omega_L^2 + c^2k_y^2}{\omega^2 - \omega_L^2 - c^2k_y^2} + 1\right) dk_y d\omega; \qquad (2.36c)$$

$$\alpha_6(x) = 2\pi^2 \iint\limits_{K<0} \left\{ c^2 |F_0|^2 + 4c^2 k_y |F_0| \frac{\partial |F_0|}{\partial k_y} + \right.$$

$$+ 2(c^2k_y^2 + \omega_L^2)\left(\frac{\partial |F_0|}{\partial k_y}\right)^2 + 2(c^2k_y^2 + \omega_L^2)\,|F_0|^2 \left(\frac{\partial \psi_0}{\partial k_y}\right)^2 +$$

$$\left. + c^4 k_y^2 |F_0|^2 \frac{1}{c^2k_y^2 + \omega_L^2 - \omega^2} \right\} \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega; \qquad (2.36d)$$

$$\alpha_7(x) = -8\pi^2 \iint\limits_{K<0} \left\{ c^2 k_y^2 |F_0|^2 + k_y |F_0| \frac{\partial |F_0|}{\partial k_y} + (c^2k_y^2 + \omega_L^2) \right\} \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega; \qquad (2.36e)$$

$$\alpha_8(x) = 2\pi^2 \iint\limits_{K<0} c^2 k_y^2 |F_0|^2 \left(\frac{c^2k_y^2 + \omega_L^2 + \omega^2}{c^2k_y^2 + \omega_L^2 - \omega^2} + 1 \right) \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega. \qquad (2.36f)$$

The values of the I_i constants and $\alpha_i(x)$ functions in expression (2.37) for the packet duration σ_{ta} are calculated in a similar way:

$$I_{11} = 2\pi^2 \iint\limits_{K\geq 0} \left\{ |F_0|^2 + 2\omega^2 \left(\frac{\partial |F_0|}{\partial \omega}\right)^2 + 4\omega |F_0| \frac{\partial |F_0|}{\partial \omega} + \right.$$

$$\left. + 2\omega^2 |F_0|^2 \left(\frac{\partial \psi_0}{\partial \omega}\right)^2 + \omega^2 |F_0|^2 \frac{1}{\omega^2 - \omega_L^2 - c^2k_y^2} \right\} dk_y d\omega; \qquad (2.37a)$$

$$I_{12} = \frac{8\pi^2}{c} \iint\limits_{K\geq 0} |F_0|^2 \frac{\partial \psi_0}{\partial \omega} \frac{\omega^3}{\sqrt{\omega^2 - \omega_L^2 - c^2k_y^2}} dk_y d\omega; \qquad (2.37b)$$

$$I_{13} = \frac{2\pi^2}{c^2} \iint\limits_{K\geq 0} \omega^2 |F_0|^2 \left(\frac{\omega^2 + \omega_L^2 + c^2k_y^2}{\omega^2 - \omega_L^2 - c^2k_y^2} + 1 \right) dk_y d\omega; \qquad (2.37c)$$

$$\alpha_9(x) = 2\pi^2 \iint\limits_{K<0} \left\{ |F_0|^2 + 2(c^2k_y^2 + \omega_L^2)\left[\left(\frac{\partial |F_0|}{\partial\omega}\right)^2 + |F_0|^2\left(\frac{\partial\psi_0}{\partial\omega}\right)^2\right] + \right.$$

$$\left. + \omega^2 |F_0|^2 \frac{1}{c^2k_y^2 + \omega_L^2 - \omega^2} \right\} \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega; \quad (2.37d)$$

$$\alpha_{10}(x) = \frac{4\pi^2}{c^2} \iint\limits_{K<0} \omega |F_0| \frac{\partial |F_0|}{\partial\omega}(c^2k_y^2 + \omega_L^2) \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega; \quad (2.37e)$$

$$\alpha_{11}(x) = \frac{2\pi^2}{c^2} \iint\limits_{K<0} \omega^2 |F_0|^2 \left(\frac{c^2k_y^2 + \omega_L^2 + \omega^2}{c^2k_y^2 + \omega_L^2 - \omega^2} + 1\right) \times$$

$$\times \exp\left\{-2x\sqrt{k_y^2 + \frac{\omega_L^2}{c^2} - \frac{\omega^2}{c^2}}\right\} dk_y d\omega. \quad (2.37f)$$

Formulas (2.36a)–(2.36f) and (2.37a)–(2.37f) are rather bulky. However, these formulas are exact expressions for arbitrary boundary conditions and represent the result in the general form. Moreover, these expressions are substantially simplified in the majority of practical cases; e.g., in the cases when angular spectra are symmetric.

§ 8. Wave packets in dispersion-free and dispersive media

From formulas (2.20)–(2.21) and (2.28) it follows that the wave packet motion in a homogeneous half-space is generally neither uniform nor rectilinear. Deviations from uniform and rectilinear motion can be observed in the dispersive and dispersion-free media. Within the scope of the spectral approach used by us, this effect is observed because inhomogeneous decaying waves exist in the two-dimensional wave packet spectrum.

Inhomogeneous waves are always present in the spectrum of any wave packet bounded along the traverse coordinate y. These waves describe purely diffraction effects in a dispersion-free medium. The presence of time dispersion makes the wave propagation mechanism much more complex; however, certain special cases can be successfully described within the scope of the general diffraction theory.

For example, medium time dispersion in a monochromatic problem leads to a change in wavelength (in phase velocity), which follows from the analysis of the wave equations presented in Chapter 1. Dispersive (1.1)–(1.2) and dispersion-free (1.6) linear wave equations for a monochromatic wave function are equivalent to the Helmholtz equation with permittivity (1.5). However, this does not mean that certain approximate approaches to solution of the wave equation in a monochromatic problem for the dispersion-free media can be used for dispersive media. For example, a formerly very popular ray approximation version, obtained as an asymptotic form at $k \to \infty$, is valid only for dispersion-free media [11]. If we apply this ray equation derivation method to the ionosphere, we obtain a paradoxical result: refraction is altogether absent in the ionosphere.

If a wave is not monochromatic, a diffraction theory cannot be used because this theory cannot adequately describe all details of the propagation process, e.g., the existence of a forerunner [1, 17].

However, the physical nature of dispersion and diffraction is more similar than it might seem at first glance because these processes can lead to similar effects (see Chapter 1).

Nevertheless, in the general case diffraction and dispersion effects have different spatial and temporal scales because, first, the energy is redistributed between homogeneous and inhomogeneous waves. In the presence of dispersion, the contribution of inhomogeneous waves increases with increasing frequency ω_L. At $\omega_L \geq \omega$, only inhomogeneous waves remain in the spectrum. Second, the contribution of different frequency components to the resultant trajectory and to the average group velocity vector is redistributed.

The scale of the effects in a dispersion-free medium is comparable with the wave packet dimensions, whereas the spatial and temporal scales of the effects in a dispersive medium depend on the intrinsic frequency of the medium ω_L and can substantially exceed the initial packet dimensions.

At large distances from the boundary, wave packet motion depends on homogeneous waves in the spectrum of boundary conditions. Packet motion asymptotically tends to uniform and linear motion with decreasing contribution of inhomogeneous waves. In the far zone, the vector of the average group velocity is constant, and the vector components are defined from (2.20)–(2.21) as

$$V_{xga} = \frac{I_1}{I_5};$$

$$V_{yga} = \frac{I_2}{I_5}.$$

The transverse coordinate (if $I_9 \neq 0$) linearly asymptotically increases at $x \to \infty$, which follows from (2.28)

$$Y_{ca} = \frac{I_8 + I_9 x}{I_5}.$$

At the initial stage, the direction of wave packet motion, on the contrary, depends on both homogeneous and inhomogeneous waves. Here, the packet motion can be substantially non-uniform. The average group velocity vector (2.20)–(2.21) changes its direction when the $\alpha_2(x)$ integral (2.21b) is nonzero. This condition is satisfied when the angular spectrum is asymmetric in the region of inhomogeneous waves ($K < 0$):

$$\left| F_0(-k_y, \omega) \right|^2 \neq \left| F_0(k_y, \omega) \right|^2.$$

Formula (2.28) describes a curvilinear wave packet propagation in the near zone if

1) $I_5 = 0$, and $I_4/I_5 \neq \alpha_4/\alpha_3$ or
2) $I_5 \neq 0$, and $\alpha_3 \neq 0$.

These conditions are satisfied when the angular spectrum is asymmetric in the region of homogeneous waves ($K \geq 0$) (in the presence of inhomogeneous waves):

$$\left| F_0 \right|^2 (-k_y, \omega) \neq \left| F_0 \right|^2 (k_y, \omega)$$

or when the $d\psi/dk_y| F_0 |^2$ product is asymmetric for decaying waves ($K < 0$)

$$\frac{\partial \psi_0}{\partial k_y}(-k_y, \omega) \left| F_0 \right|^2 (-k_y, \omega) \neq \frac{\partial \psi_0}{\partial k_y}(k_y, \omega) \left| F_0 \right|^2 (k_y, \omega).$$

Thus, wave packet motion in a homogeneous medium becomes curvilinear when the angular spectrum of the boundary conditions is asymmetric. During propagation, a "bounded" structure gradually becomes symmetrized, and the lateral shift disappears.

At first glance, the obtained results completely contradict to many works (e.g., [19, 21]), where it is stated that the wave packet energy center moves uniformly and linearly in dispersive or dispersion-free homogeneous media.

This illusory contradiction is eliminated at a detailed consideration of the Noether theorem, according to which field moments are conserved if the Lagrangian operator is invariant with respect to shift and rotation [49, 78, 82, 83]. Such a mathematical abstraction is asymptotically valid for the spatial regions where a wave packet ceases "sensing" a source or boundary that generated this packet. From the spectrum viewpoint, this means that inhomogeneous waves are absent in the angular spectrum.

In this chapter we examined a more general problem with the boundary, the presence of which disturbs system invariance with respect to shift and rotation. Our results in the far zone naturally coincide with the classical results obtained using the Noether theorem.

A "spinner", well-known in footbal or tennis, can be a certain mechanical analog of a controlled nonlinear motion in a homogeneous medium (in air). For example, a well-known "Olympic goal" makes it possible to send spinner-ball from a corner to a goalmouth.

We emphasize once more that the necessary condition for nonlinear propagation is the presence of inhomogeneous (decaying) waves in the packet angular spectrum.

When choosing the appropriate spectrum and, correspondingly, the spatial and temporal field distribtuion at the boundary, we can arbitrarily specify the unintial and final wave packet directions. A quasi-monochromatic packet with transverse frequency modulation can be an example of such a packet moving along a nonlinear trajectory.

We now compare the results of the numerical computation of packet motion in a homogeneous three-dimensional medium with and without time dispersion in order to help the reader to better understand the sense of transverse frequency modulation and packet transverse shift due to time dispersion.

The wave field in the $x \geq 0$ halft-space was obtained using the numerical methods based on the integral representation [106]:

$$U(x,y,z,t) = \iiint_{-\infty}^{\infty} g(x,y-\eta,z-\xi,t-\tau)\,U(0,\eta,\xi,\tau)\,d\eta\,d\xi\,d\tau.$$

Here,

$$g(x,y,z,t) = \frac{\delta\left(t-\sqrt{x^2+y^2+z^2}/c\right)}{\sqrt{x^2+y^2+z^2}} -$$

$$-\frac{\omega_L}{\sqrt{t^2-(x^2+y^2+z^2)/c^2}}J_1\left(\omega_L\sqrt{t^2-(x^2+y^2+z^2)/c^2}\right)$$

is the Green's function for Eq. (1.7) in the $t^2 \geq (x^2 + y^2 + z^2)/c^2$ region. Outside this region, $g(x, y, z, t) = 0$.

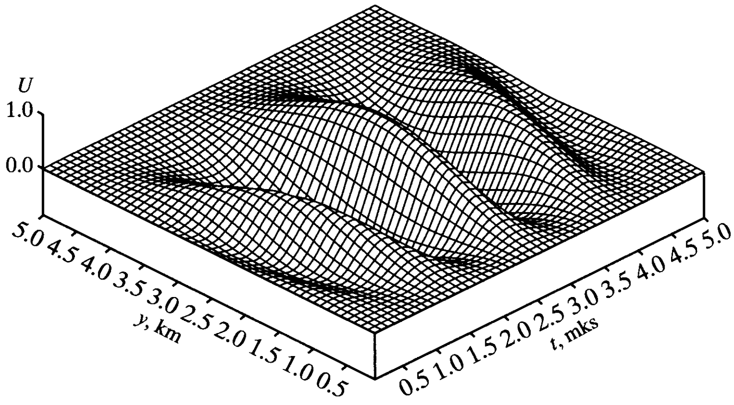

Fig. 1. Field function $U(t, y)$ at the $x = 0$ boundary

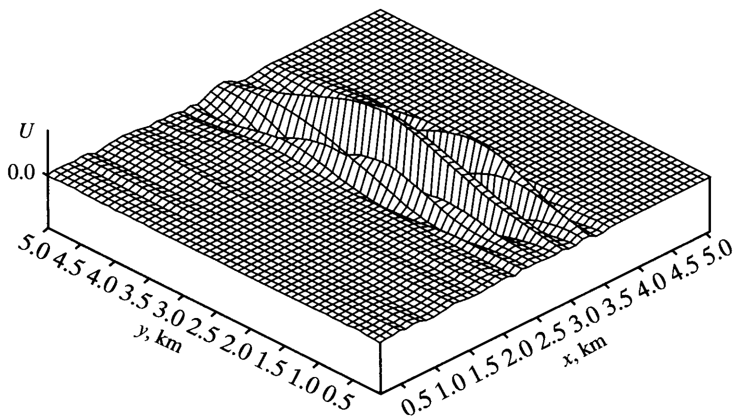

Fig. 2. Field function $U(x, y)$ in a dispersion-free medium at instant $t = 12$ mks

We selected the special type of boundary conditions localized in the $z = 0$ plane in order to decrease the number of variables and simplify perception of results: $U(0, y, z, t) = U(y, t)\,\delta(z)$.

Figure 1 shows the version of the $U(y,t)$ field function at the boundary, which specifies the packet with transverse frequency modulation. We selected such a function with only several oscillations because we tried to schematically demonstrate a difference in the wave packet behavior during packet propagation in homogeneous media without and with time dispersion.

The central frequency of this broadband wave function linearly varies from 0.5 (at $y = 5.0$ km) to 1 MHz (at $y = 0$ km).

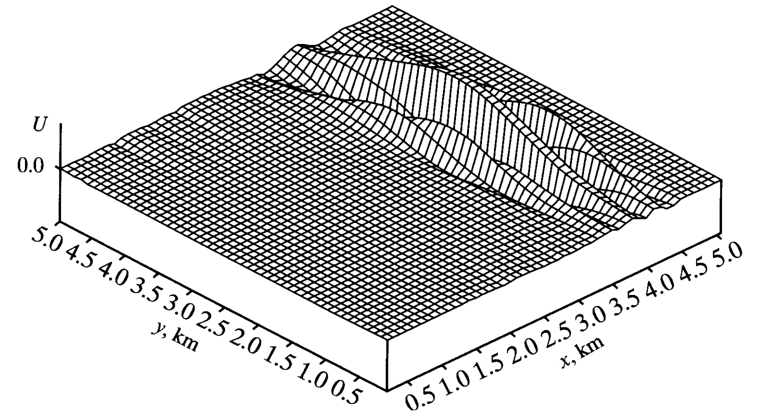

Fig. 3. Field function $U(x, y)$ in a dispersion-free medium at t = 15 mks

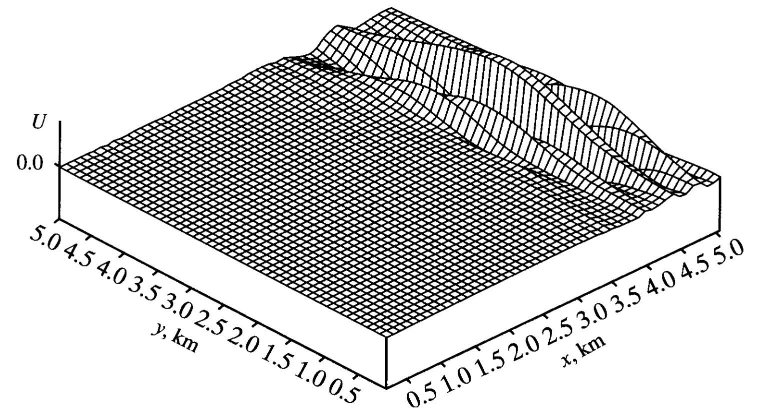

Fig. 4. Field function $U(x, y)$ in a dispersion-free medium at t = 18 mks

To simplify the visualization, we specify $z = 0$ and represent the series of field function distributions $U(x, y, 0, t)$ in the (x, y) plane at different instants t.

Figures 2–4 show the field distribution in a dispersion-free medium ($\omega_L = 0$) at instants t = 12, 15, and 18 mks. It is clear that a wave packet in a dispersion-free medium moves along the linear trajectory at the velocity of light c without a visible change in the packet waveform.

Figures 5–9 demonstrate a similar field distribution in a homogeneous medium with time dispersion at f_L = 0.6 MHz ($f_L = \omega_L/2\pi$) at instants t = 12, 15, 18, 21, and 24 mks. The formation and propagation of a forerunner at the velocity c can be observed in Figs. 5–7.

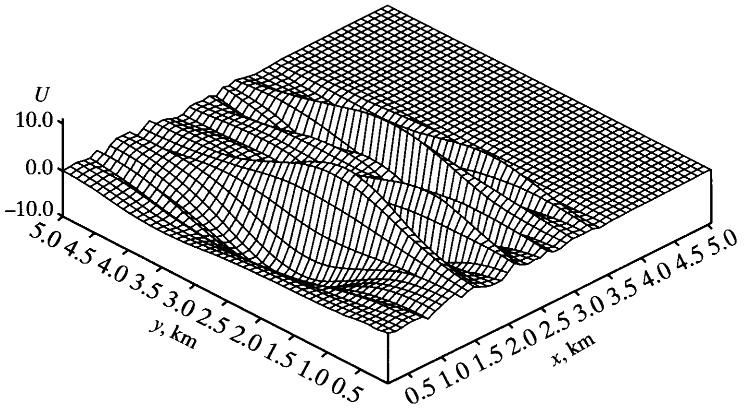

Fig. 5. Field function $U(x, y)$ in a dispersive medium at $t = 12$ mks

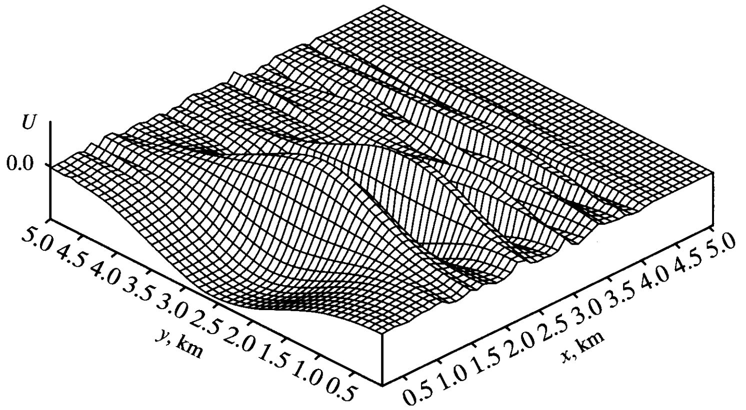

Fig. 6. Field function $U(x, y)$ in a dispersive medium at $t = 15$ mks

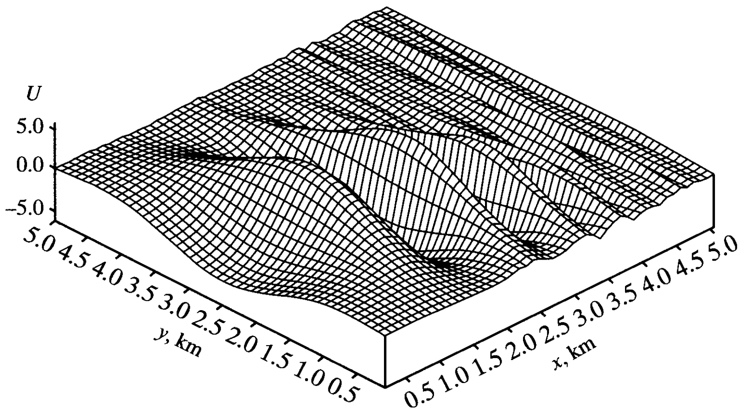

Fig. 7. Field function $U(x, y)$ in a dispersive medium at $t = 18$ mks

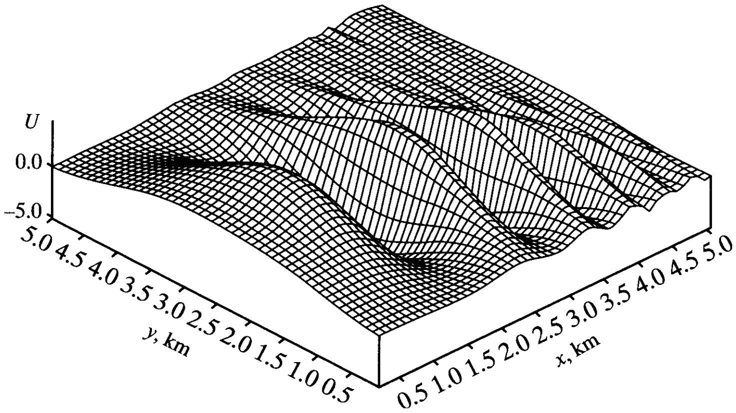

Fig. 8. Field function $U(x, y)$ in a dispersive medium at $t = 21$ mks

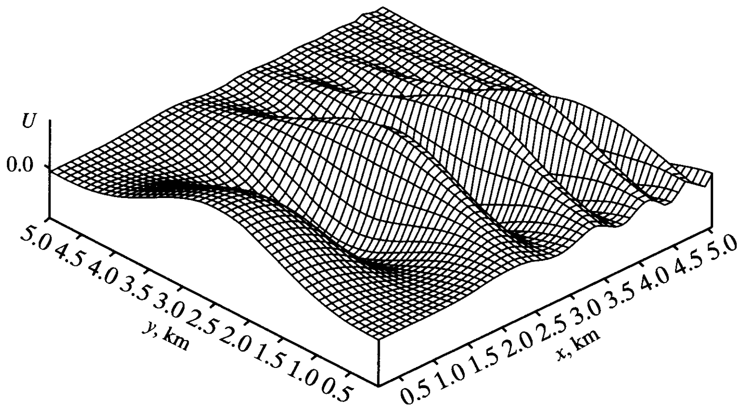

Fig. 9. Field function $U(x, y)$ in a dispersive medium at $t = 24$ mks

The form of the main wave packet body undergoes dispersive distortions, which are shown as a body longitudinal extension and a visible shift to the right. This indicates that the packet energy center moves nonlinearly.

§ 9. Discussion of results

We considered the problem of wave packet propagation in a homogeneous half-space with time dispersion specified by KGE (1.7) and obtained the exact analytical expressions for the integral energy characteristics of these packets: the average group velocity vector (2.4), transverse coordinate (2.5), propagation time (2.6), width (2.7), and duration (2.8).

These expressions were obtained in the general form for arbitrary initial conditions $U(0, y, t)$ at the $x = 0$ boundary.

We found out that time dispersion of a medium can lead to a change in the wave packet direction in a homogeneous medium under certain conditions. This effect is observed against a background of diffraction phenomena that can also contribute to the lateral shift of a packet. However, the spatial and temporal scales of these phenomena are different, which makes it possible to separate the phenomena.

The considered effect of the packet transverse shift due to time dispersion and the dispersion refraction effect discussed in the Introduction are of the same physical nature characterized by the nonlinear dependence of the wavenumber k on frequency ω in the dispersion equation.

However, the packet transverse shift is the sufficient but not necessary condition of existence of the dispersion refraction effect of the local (RO) character. The general considerations indicate that a change in the direction of isolated wave packet parts not always leads to a change in the direction of motion of the energy center, i.e., the entire packet.

Here, the situation is as in the case of a mechanical analog of rocket motion in vacuum: a rocket can perform arbitrary complex maneuvers, but the rocket center of mass and consumed fuel level will remain invariable (or will move uniformly and linearly).