Chapter 5

Modified version of space-time RO for media with an arbitrary time dispersion

§ 1. Introduction

In this chapter, we develop the modified STRO version for the media with an arbitrary time dispersion, which is the main subject of this monograph.

The development is based on the derivation of the STRO equations considered in the previous chapter. The results previously obtained for the particular case of time dispersion specified by KGE, namely the effect of dispersion refraction and the correction of the ordinary refraction value, which cannot be described within the scope of the standard STRO approach, should evidently be valid for all media with time dispersion because these effects are caused by general nonlinear variations in frequency ω and wavenumber k in the dispersion equation.

§ 2. Eikonal and transport equations

To obtain the eikonal and transport equations, we substitute Anzatz

$$U(\mathbf{r},t) = A(\mathbf{r},t)\exp\{i\Psi(\mathbf{r},t)\}$$

into (1.1a)–(1.2a) and separate the real and imaginary parts of the complex equation

$$\nabla^2 A - \frac{\varepsilon_0}{c^2}\frac{\partial^2 A}{\partial t^2} - A\mathbf{k}^2 + A\frac{\varepsilon_0}{c^2}\omega^2 - A\frac{\omega_M}{c^2} + \frac{1}{2c^2}\frac{\partial^2\omega_M}{\partial\omega^2}\frac{\partial^2 A}{\partial t^2} +$$
$$+\frac{i}{c^2}\frac{\partial\omega_M}{\partial\omega}\frac{\partial A}{\partial t} + i\nabla\cdot\mathbf{k}A + \frac{i\varepsilon_0}{c^2}\frac{\partial\omega}{\partial t}A + i2\mathbf{k}\cdot\nabla A +$$
$$+ i\frac{2\varepsilon_0}{c^2}\omega\frac{\partial A}{\partial t} = 0. \qquad (5.1)$$

Here (as before) the gradient $\nabla\Psi$ is a local wavevector $\mathbf{k}(\mathbf{r}, t)$, and $-\partial\Psi/\partial t$ is a local frequency $\omega(\mathbf{r}, t)$.

Integral operator $M(U)$ (1.2a) is reduced to the differential operators in the following way.

We expand the Anzatz into the Taylor series

$$U=\left(A_0+\frac{\partial A}{\partial t}t+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2\right)\exp\{i(\psi_0-\omega t)\}.$$

After the substitution of this expression into (1.2a), we have for $t=0$

$$M(U)=\mathrm{e}^{i\psi_0}\int_0^\infty\frac{\partial^2 h}{\partial t^2}(\tau)\left[A_0-\frac{\partial A}{\partial t}\tau+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}\tau^2\right]\exp(-i\omega\tau)d\tau. \qquad (5.2)$$

Note, that the integral expression (5.2) is equivalent to the Fourier transform (1.5). Using the obvious properties of the Fourier transform

$$\frac{1}{2\pi}\int_0^\infty\tau^m\tilde{\varepsilon}(\mathbf{r},\tau)\exp(i\omega\tau)d\tau=(-i)^m\frac{\partial^m\varepsilon(\mathbf{r},\omega)}{\partial\omega^m}$$

we obtain

$$M(U)=\mathrm{e}^{i\psi_0}\left[\omega^2(\varepsilon_0-\varepsilon)A_0-i\frac{\partial(\omega^2(\varepsilon_0-\varepsilon))}{\partial\omega}\frac{\partial A}{\partial t}-\right.$$

$$\left.-\frac{1}{2}\frac{\partial^2(\omega^2(\varepsilon_0-\varepsilon))}{\partial\omega^2}\frac{\partial^2 A}{\partial t^2}\right].$$

Here we neglected the $-i/2(d\omega/dt)t^2$ term of the phase function ψ, because its effect on the results is insignificant due to the STRO applicability conditions (4.1).

Actually, d^2h/dt^2 is a "fast" function as compared to

$$\exp\left(-\frac{i}{2}\frac{\partial\omega}{\partial t}t^2\right);$$

consequently,

$$\frac{\partial^2 h}{\partial t^2}(\tau)\exp\left(-\frac{i}{2}\frac{\partial\omega}{\partial t}\tau^2\right)\approx\frac{\partial^2 h}{\partial t^2}(\tau).$$

In the asymptotic expression for $M(U)$, the integration process, which is essentially responsible for wave "memory" of previous values (i.e., time dispersion), is implicitly present in the ε value (1.5).

The real part of (5.1) contains the eikonal equation

$$\mathbf{k}^2-\frac{\varepsilon_0\omega^2}{c^2}+\frac{\omega_M}{c^2}=0, \qquad (5.3)$$

where

$$\omega_M = \omega^2(\varepsilon_0 - \varepsilon)$$

and the additional terms are defined as

$$\nabla^2 A - \frac{\varepsilon_0}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{1}{2c^2}\frac{\partial^2 \omega_M}{\partial \omega^2}\frac{\partial^2 A}{\partial t^2}.$$

The imaginary part contains the transport equation

$$\nabla \cdot \mathbf{k}A + \frac{\varepsilon_0}{c^2}\frac{\partial \omega}{\partial t}A + 2\mathbf{k}\cdot\nabla A + \frac{2\varepsilon_0}{c^2}\omega\frac{\partial A}{\partial t} = 0 \qquad (5.4)$$

and the additional term

$$\frac{i}{c^2}\frac{\partial \omega_M}{\partial \omega}\frac{\partial A}{\partial t}.$$

As compared to KGE, which also contains additional terms of the eikonal equation

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$$

of the order of smallness $O(\chi^2)$, one more term

$$\frac{1}{2c^2}\frac{\partial^2 \omega_M}{\partial \omega^2}\frac{\partial^2 A}{\partial t^2}$$

of the order of smallness $O(\chi^4)$ originates in the general case.

For an arbitrary dispersion law, the term

$$\frac{1}{c^2}\frac{\partial \omega_M}{\partial \omega}\frac{\partial A}{\partial t}$$

of the order of smallness $O(\chi^2)$, which is absent in KGE, appears in the imaginary part of (5.1) (i.e., in the transport equation).

§ 3. Standard STRO version

Below we will repeat the calculations performed in the previous chapter for KGE, but for the case of an arbitrary dispersion law. The group velocity vector

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_g \qquad (5.5)$$

specifies the relationships between the derivatives

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla. \qquad (5.6)$$

Writing $\mathbf{V}_g = d\omega/d\mathbf{k}$, we obtain the following relationships from the dispersion law (5.3):

$$\mathbf{V}_g = \frac{d\omega}{d\mathbf{k}} = c^2 \frac{\mathbf{k}}{\varepsilon_0 \omega}. \tag{5.7}$$

Differentiating (5.7), we obtain derivative $d\mathbf{V}_g/dt$, which describes refraction effects:

$$\frac{d\mathbf{V}_g}{dt} = \frac{c^2}{\varepsilon_0 \omega}\frac{d\mathbf{k}}{dt} - \frac{c^2\mathbf{k}}{\varepsilon_0^2 \omega}\frac{d\varepsilon_0}{dt} - \frac{c^2\mathbf{k}}{\varepsilon_0 \omega^2}\frac{d\omega}{dt}.$$

Differentiation of (5.3) with respect to **r** and t determines the relationship between the partial derivatives:

$$(\mathbf{k}\times\nabla)\mathbf{k} + \mathbf{k}\cdot\times(\nabla\times\mathbf{k}) = \frac{\varepsilon_0\omega\nabla\omega}{c^2} + \frac{\omega^2\nabla\varepsilon_0}{2c^2} - \frac{\nabla\omega_M}{2c^2} \tag{5.8}$$

and

$$\frac{\partial\omega}{\partial t} = \frac{c^2\mathbf{k}}{\varepsilon\omega}\cdot\frac{\partial\mathbf{k}}{\partial t} - \frac{\omega}{2\varepsilon_0}\frac{\partial\varepsilon_0}{\partial t} + \frac{1}{2\omega\varepsilon_0}\frac{\partial\omega_M}{\partial t} =$$

$$= -\frac{c^2\mathbf{k}}{\varepsilon_0\omega}\cdot\nabla\omega - \frac{\omega}{2\varepsilon_0}\frac{\partial\varepsilon_0}{\partial t} + \frac{1}{2\omega\varepsilon_0}\frac{\partial\omega_M}{\partial t}. \tag{5.9}$$

In a standard approach, $\mathbf{k}\times(\nabla\times\mathbf{k})$ in (5.8) is zero, which is equivalent to the introduction of a locally plane monochromatic homogeneous wave as a field model.

For a stationary medium, where $\partial\varepsilon_0/\partial t = 0$, $\partial\omega_M/\partial t = 0$, we obtain

$$\frac{d\omega}{dt} = 0.$$

Indeed, in accordance with (5.6),

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + \mathbf{V}_g\cdot\nabla\omega.$$

On the other hand, for a stationary medium we obtain from (5.7) and (5.9)

$$\frac{d\omega}{dt} = -\mathbf{V}_g\cdot\nabla\omega.$$

The expression for the $d\mathbf{k}/dt$ derivative follows from (5.6) and (5.8)

$$\frac{d\mathbf{k}}{dt} = \frac{\partial\mathbf{k}}{\partial t} + (\mathbf{V}_g\cdot\nabla)\mathbf{k} = \frac{\omega\nabla\varepsilon_0}{2\varepsilon_0} - \frac{\nabla\omega_M}{2\varepsilon_0\omega}.$$

Finally, the group velocity vector derivative is written as

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2}{2\varepsilon_0^2}\left(1 - \frac{\omega_M}{\varepsilon_0\omega^2}\right)\nabla\varepsilon_0 - \frac{c^2}{2\varepsilon_0^2\omega^2}\nabla\omega_M. \tag{5.10}$$

In a homogeneous medium ($\nabla\varepsilon_0 = 0$, $\nabla\omega_M = 0$) the derivative is defined as

$$\frac{d\mathbf{V}_g}{dt} = 0,$$

which means that it is impossible to describe the dispersive refraction effect using the standard STRO version.

§ 4. Alternative method for deriving ray equations

As before, we now use the group velocity vector definition

$$\mathbf{V}_g = \frac{\langle \mathbf{P} \rangle}{\langle W \rangle}. \tag{5.11}$$

We consider a stationary inhomogeneous medium. Assume that the amplitude and phase of field U at point $\mathbf{R}_0$, T_0 and in the vicinity of this point are specified by the Taylor series:

$$U(\mathbf{R}_0 + \mathbf{r}, T_0 + t) = \left\{ A_0 + \mathbf{r}\cdot\nabla A + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A + \frac{\partial A}{\partial t}t + \frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2 + \right.$$
$$\left. + \mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)t \right\}\exp\left\{ i\left(\Psi_0 + \mathbf{r}\cdot\mathbf{k} + \frac{1}{2}\mathbf{r}\cdot(\mathbf{r}\cdot\nabla)\mathbf{k} - \omega t - \right.\right.$$
$$\left.\left. - \frac{1}{2}\frac{\partial\omega}{\partial t}t^2 - \mathbf{r}\cdot\nabla\omega t\right)\right\}. \tag{5.12}$$

Then, using the technique described in Chapter 4, we find gradient ∇U and derivative dU/dt for field (5.12) and calculate the $\mathbf{P}$ and W values from formulas (1.13)–(1.14).

Assume that the STRM applicability conditions (4.1) are satisfied. For media with an arbitrary dispersion law, the applicability conditions are specified as

$$\frac{\nabla\cdot\mathbf{k}}{|\mathbf{k}|} \sim \frac{|\nabla\times\mathbf{k}|}{|\mathbf{k}|} \sim \frac{|\nabla\omega|}{\omega} \sim \frac{\nabla^2 A}{|\nabla A|} \sim \frac{|\nabla A|}{A} \sim \frac{1}{L_W};$$

$$\frac{|\nabla\varepsilon_0|}{\varepsilon_0} \sim \frac{|\nabla\omega_M|}{\omega_M} \sim \frac{1}{L_P};$$

$$\frac{\partial\omega/\partial t}{\omega} \sim \frac{\partial^2 A/\partial t^2}{\partial A/\partial t} \sim \frac{\partial A/\partial t}{A} \sim \frac{1}{T_W};$$

$$\lambda = \frac{2\pi}{|\mathbf{k}|}; \quad \tau = \frac{2\pi}{\omega}; \quad \frac{\lambda}{L_W} \sim \frac{\lambda}{L_P} \sim \frac{\tau}{T_W} \sim \chi \ll 1.$$

We now find the average energy flux $\mathbf{P}$ and density W at the observational point $\mathbf{R}_0$, T_0 and in the vicinity of this point:

$$\langle \mathbf{P} \rangle = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \mathbf{P}(\mathbf{r}, t+\xi) d\xi; \quad \langle W \rangle = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} W(\mathbf{r}, t+\xi) d\xi.$$

By integrating the expressions for $\mathbf{P}$ and W during the period $\tau = 2\pi/(\omega + \mathbf{r}\cdot\nabla\omega)$ we obtain

$$\langle \mathbf{P} \rangle = \frac{c^2}{2\tau}\left\{-\left(\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial t^2}t + \mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)\right)\times\right.$$
$$\times\left(\nabla A + (\mathbf{r}\cdot\nabla)\nabla A + \nabla\frac{\partial A}{\partial t}t\right) + \left(A_0 + \mathbf{r}\cdot\nabla A + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A + \frac{\partial A}{\partial t}t + \right.$$
$$\left. + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\times$$
$$\left.\times\,\omega + \mathbf{r}\cdot\nabla\omega + \frac{\partial\omega}{\partial t}t(\mathbf{k} + (\mathbf{r}\cdot\nabla)\mathbf{k} - \nabla\omega t)\right\} \tag{5.13}$$

and

$$\langle W \rangle = \frac{1}{4\tau}\left\{\left(\varepsilon_0 + \mathbf{r}\cdot\nabla\varepsilon_0 + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\varepsilon_0\right)\left(\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial t^2}t + \mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)\right)^2 + \right.$$
$$\left. + c^2\left(\nabla A + (\mathbf{r}\cdot\nabla)\nabla A + \left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\right\} + \frac{1}{4\tau}\left(A_0 + \mathbf{r}\cdot\nabla A + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2 A + \right.$$
$$\left. + \frac{\partial A}{\partial t}t + \frac{1}{2}\frac{\partial^2 A}{\partial t^2}t^2 + \mathbf{r}\cdot\left(\nabla\frac{\partial A}{\partial t}\right)t\right)^2\left\{\left(\varepsilon_0 + \mathbf{r}\cdot\nabla\varepsilon_0 + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\varepsilon_0\right)\times\right.$$
$$\times\left(\omega + \mathbf{r}\cdot\nabla\omega + \frac{\partial\omega}{\partial t}t\right)^2 + c^2(\mathbf{k} + (\mathbf{r}\cdot\nabla)\mathbf{k} - \nabla\omega t)^2 +$$
$$\left. + \left(\omega_M + \mathbf{r}\cdot\nabla\omega_M + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2\omega_M\right)\right\}. \tag{5.14}$$

Here, we neglected the high-order terms that appeared during the integration.

The group velocity vector at point $\mathbf{R}_0$, T_0 and in the vicinity of this point is expressed by formulas (5.11), (5.13) and (5.14).

The vector has the following form at point $\mathbf{R}_0$, T_0:

$$\mathbf{V}_g = c^2\frac{\omega\mathbf{k} - (\partial A/\partial t)\nabla A/A_0^2}{\varepsilon_0\omega^2 + 0.5\,\{\varepsilon_0(\partial A/\partial t)^2 + c^2(\nabla A)^2\}/A_0^2}. \tag{5.15}$$

If we reject the terms of the order of $O(\chi^2)$ in (5.15), we obtain the formula coincident with (5.7):

$$\mathbf{V}_g = c^2 \frac{\mathbf{k}}{\varepsilon_0 \omega}.$$

Choosing t as an independent variable and taking into account that

$$\frac{dt}{dt} = 1; \quad \frac{d\mathbf{r}}{dt} = \mathbf{V}_g = c^2 \frac{\mathbf{k}}{\varepsilon_0 \omega},$$

we find the derivative of the group velocity vector:

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2}{2\varepsilon_0^2}\left(1 - \frac{\omega_M}{\varepsilon_0 \omega^2}\right)\nabla\varepsilon_0 - \frac{c^2}{2\varepsilon_0^2 \omega^2}\left(1 - \frac{\omega_M}{\varepsilon_0 \omega^2}\right)\nabla\omega_M +$$
$$+ \frac{c^4 \omega_M}{\varepsilon_0^3 \omega^4}(\mathbf{k}\cdot\nabla)\mathbf{k} - \frac{c^2 \omega_M}{\varepsilon_0^2 \omega^3}\nabla\omega. \qquad (5.16)$$

Here we neglected the terms of the order of $O(\chi^2)$.

§ 5. Field models

Let us begin with the standard field model. We consider the general case of a stationary inhomogeneous medium for the wave with transverse and longitudinal frequency modulation.

Assume that vector $\mathbf{k}$ at point X_0, Y_0, T_0 is directed along the x axis. This direction was selected in order to simplify the formulas and does not affect the generality of results. We define the center frequency ω, frequency gradient (modulation) $\nabla\omega = \mathbf{e}_x(\partial\omega/\partial x) + \mathbf{e}_y(\partial\omega/\partial y)$, derivative dk_y/dy (specifying ray divergence), and the medium parameters ε_0, ω_M and

$$\nabla\varepsilon_0 = \mathbf{e}_x(\partial\varepsilon_0/\partial x) + \mathbf{e}_y(\partial\varepsilon_0/\partial y), \quad \nabla\omega_M = \mathbf{e}_x(\partial\omega_M/\partial x) + \mathbf{e}_y(\partial\omega_M/\partial y).$$

In the standard STRO version, the remaining field parameters are determined from Eqs. (5.3), (5.8), and (5.9). Thus, the value of the longitudinal wave vector k_x follows from (5.3):

$$k_x = \sqrt{\frac{\varepsilon_0 \omega^2}{c^2} - \frac{\omega_M}{c^2}}.$$

The derivatives of vector $\mathbf{k}$ are determined from (5.8):

$$\frac{\partial\mathbf{k}}{\partial x} = \mathbf{e}_x\left(\frac{\varepsilon_0\omega}{c^2 k_x}\frac{\partial\omega}{\partial x} + \frac{\omega^2}{2c^2 k_x}\frac{\partial\varepsilon_0}{\partial x} - \frac{1}{2c^2 k_x}\frac{\partial\omega_M}{\partial x}\right) +$$
$$+ \mathbf{e}_y\left(\frac{\varepsilon_0\omega}{c^2 k_x}\frac{\partial\omega}{\partial y} + \frac{\omega^2}{2c^2 k_x}\frac{\partial\varepsilon_0}{\partial y} - \frac{1}{2c^2 k_x}\frac{\partial\omega_M}{\partial y}\right).$$

It follows from the condition $\nabla \times \mathbf{k} = 0$ that $\partial k_x/\partial y = \partial k_y/\partial x$:

$$\frac{\partial \mathbf{k}}{\partial y} = \mathbf{e}_x\left(\frac{\varepsilon_0\omega}{c^2 k_x} - \frac{\partial\omega}{\partial y} + \frac{\omega^2}{2c^2 k_x}\frac{\partial\varepsilon_0}{\partial y} - \frac{1}{2c^2 k_x}\frac{\partial\omega_M}{\partial y}\right) + \mathbf{e}_y\frac{\partial k_y}{\partial y}.$$

Equation (5.9) defines the derivative $\partial\omega/\partial t$:

$$\frac{\partial\omega}{\partial t} = -\frac{c^2 k_x}{\varepsilon_0\omega}\frac{\partial\omega}{\partial x}.$$

The Taylor expansion of the phase function Ψ at $X_0 + x$, $Y_0 + y$, $T_0 + t$ is written as

$$\Psi = \Psi_0 + k_x x + \frac{1}{2}\frac{\partial k_x}{\partial x}x^2 + \frac{\partial k_y}{\partial x}xy + \frac{\partial k_x}{\partial y}yx + \frac{1}{2}\frac{\partial k_y}{\partial y}y^2 -$$

$$-\omega t - \frac{\partial\omega}{\partial x}xt - \frac{\partial\omega}{\partial y}yt - \frac{1}{2}\frac{\partial\omega}{\partial t}t^2. \tag{5.17}$$

Amplitude function A is the linear function of x and t:

$$A = A_0 + \frac{\partial A}{\partial x}x + \frac{\partial A}{\partial t}t. \tag{5.18}$$

We determine A_0 and one of its derivatives at point X_0, Y_0, T_0. Determining, e.g., $\partial A/\partial x$, we find the other derivative dA/dt from the transport equation (5.4):

$$\frac{\partial A}{\partial t} = -\frac{c^2}{2\varepsilon_0\omega}\left(\frac{\partial k_x}{\partial x} + \frac{\partial k_y}{\partial y} + \frac{\varepsilon_0}{c^2}\frac{\partial\omega}{\partial t}\right)A_0 - \frac{c^2 k_x}{\varepsilon_0\omega}\frac{\partial A}{\partial x}. \tag{5.19}$$

We now estimate the error of this model by substituting the field function $U = A\exp(i\Psi)$, where A and Ψ are specified by expressions (5.17) and (5.18), into Eqs. (1.1a)–(1.2a) and obtain the discrepancy equation.

Analyzing a difference between the exact and model solutions, we find out that the maximum error in the field description is defined by the linear term with y:

$$-2\left(\frac{\varepsilon_0\omega}{c^2}\frac{\partial\omega}{\partial y} + \frac{\omega^2}{2c^2}\frac{\partial\varepsilon_0}{\partial y} - \frac{1}{2c^2}\frac{\partial\omega_M}{\partial y}\right)Ay = 0.$$

We now modify the standard model by introducing the smooth transverse function of amplitude $A_1(y)$, which compensates the systematic error in STRO. This function is described by the Airy equation

$$\frac{\partial^2 A_1}{\partial y^2} - 2\left(\frac{\varepsilon_0\omega}{c^2}\frac{\partial\omega}{\partial y} + \frac{\omega^2}{2c^2}\frac{\partial\varepsilon_0}{\partial y} - \frac{1}{2c^2}\frac{\partial\omega_M}{\partial y}\right)A_1 y = 0.$$

It is necessary to exclude derivative $\partial k_y/\partial x$ from the phase function (5.17) in order to transform the standard model into the modified one. The phase function Ψ_1 takes the following form in a new model:

$$\Psi_1 = \Psi_0 + k_x x + \frac{1}{2}\frac{\partial k_x}{\partial x}x^2 + \frac{\partial k_x}{\partial y}yx + \frac{1}{2}\frac{\partial k_y}{\partial y}y^2 -$$

$$-\,\omega t - \frac{\partial \omega}{\partial x}xt - \frac{\partial \omega}{\partial y}yt - \frac{1}{2}\frac{\partial \omega}{\partial t}t^2. \qquad (5.20)$$

The remaining relationships between the phase partial derivates are the same:

$$\frac{\partial k_x}{\partial x} = \frac{\varepsilon_0 \omega}{c^2 k_x}\frac{\partial \omega}{\partial x} + \frac{\omega^2}{2c^2 k_x}\frac{\partial \varepsilon_0}{\partial x} - \frac{1}{2c^2 k_x}\frac{\partial \omega_M}{\partial x}, \qquad (5.8a)$$

$$\frac{\partial k_x}{\partial y} = \frac{\varepsilon_0 \omega}{c^2 k_x}\frac{\partial \omega}{\partial y} + \frac{\omega^2}{2c^2 k_x}\frac{\partial \varepsilon_0}{\partial y} - \frac{1}{2c^2 k_x}\frac{\partial \omega_M}{\partial y}, \qquad (5.8b)$$

$$\frac{\partial \omega}{\partial t} = -\frac{c^2 k_x}{\varepsilon_0 \omega}\frac{\partial \omega}{\partial x}. \qquad (5.9a)$$

As in the standard model, $d\omega/dt = 0$ in a new field model since it follows from (5.6) that

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \mathbf{V}_g\frac{\partial \omega}{\partial x},$$

and derivative $\partial\omega/\partial t$ in (5.9a) is expressed as

$$\frac{\partial \omega}{\partial t} = -\,\mathbf{V}_g\frac{\partial \omega}{\partial x}.$$

We can obtain the group velocity vector derivative for the new model from (5.16) by substituting the relationships between the derivatives (5.8a) and (5.8b) for the phase function (5.20) into this expression:

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2}{2\varepsilon_0^2}\nabla\varepsilon_0 + \frac{c^2\omega_M}{2\varepsilon_0^3\omega^2}\nabla_\perp\varepsilon_0 - \frac{c^2}{2\varepsilon_0^2\omega^2}\nabla\omega_M +$$

$$+\frac{c^2\omega_M}{2\varepsilon_0^3\omega^4}\nabla_\perp\omega_M - \frac{c^2\omega_M}{2\varepsilon_0^2\omega^3}\nabla_\perp\omega.$$

Here the last term in the right side describes the dispersive refraction effect, the value of which depends on the transverse frequency modulation index $\nabla_\perp\omega$ and dispersion properties of the medium that are characterized by the ω_M parameter.

A comparison with formula (5.10) shows that a new field model corrects the regular refraction magnitude.

The total derivative of amplitude A can be obtained directly from the transport equation (5.4), taking into account relationship (5.6) between the derivatives

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{c^2 k_x}{\varepsilon_0 \omega} \frac{\partial A}{\partial x}.$$

Substituting the partial derivative values (5.8a) and (5.9a) into the transport equation, we obtain

$$\frac{dA}{dt} = -\frac{A_0}{2}\left[\frac{c^2}{\varepsilon_0 \omega}\frac{\partial k_y}{\partial y} + \left(\frac{1}{k_x} - \frac{c^2 k_x}{\varepsilon_0^2 \omega^2}\right)\frac{\partial \omega}{\partial x} + \right.$$
$$\left. + \frac{\omega}{2\varepsilon_0 k_x}\frac{\partial \varepsilon_0}{\partial x} - \frac{1}{2\varepsilon_0 \omega k_x}\frac{\partial \omega_M}{\partial x}\right].$$

We now write all ray equations obtained for the new field model:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}_g;$$

$$\mathbf{V}_g = c^2 \frac{\mathbf{k}}{\varepsilon_0 \omega};$$

$$\frac{d\omega}{dt} = 0; \qquad (5.21)$$

$$\frac{d\mathbf{V}_g}{dt} = -\frac{c^2}{2\varepsilon_0^2}\nabla \varepsilon_0 + \frac{c^2 \omega_M}{2\varepsilon_0^3 \omega^2}\nabla_\perp \varepsilon_0 - \frac{c^2}{2\varepsilon_0^2 \omega^2}\nabla \omega_M +$$
$$+ \frac{c^2 \omega_M}{2\varepsilon_0^3 \omega^4}\nabla_\perp \omega_M - \frac{c^2 \omega_M}{\varepsilon_0^2 \omega^3}\nabla_\perp \omega;$$

$$\frac{dA}{dt} = -\frac{A_0}{2}\left[\frac{c^2}{\varepsilon_0 \omega}\frac{\partial k_y}{\partial y} + \left(\frac{1}{k_x} - \frac{c^2 k_x}{\varepsilon_0^2 \omega^2}\right)\frac{\partial \omega}{\partial x} + \right.$$
$$\left. + \frac{\omega}{2\varepsilon_0 k_x}\frac{\partial \varepsilon_0}{\partial x} - \frac{1}{2\varepsilon_0 \omega k_x}\frac{\partial \omega_M}{\partial x}\right].$$

The set of equations (5.21) is closed only for monochromatic waves. This system is generally not closed for modulated waves; i.e., it cannot be used to trace rays based only on initial conditions and medium parameters since the total derivative $d(\partial\omega/\partial y)/dt$ along the ray, which describes a variation in transverse frequency modulation, is not defined. To calculate the amplitude function, it is also necessary to know total derivatives $d(\partial\omega/\partial y)/dt$ and $d(\partial k_y/\partial y)/dt$.

One of possible methods for calculating these derivatives will be considered in the next paragraph.

§ 6. Quasi-ray field model

To simplify the calculations, we introduce the following designations:

$$K_x = \frac{\partial k_x}{\partial x};\ K_y = \frac{\partial k_x}{\partial y};\ D = \frac{\partial k_y}{\partial y};\ \Omega_x = -\frac{\partial \omega}{\partial x};\ \Omega_y = -\frac{\partial \omega}{\partial y};\ \Omega_t = \frac{\partial \omega}{\partial t}.$$

Having introduced the new field model through the phase function (5.20), we obtained that the exact and modeled solution coincide with the quadratic terms in the real part and with the linear terms in the imaginary part of the discrepancy equation.

Modifying this model, we can also take into account these terms by considering the derivatives of K_x, K_y, D, Ω_x, Ω_y, and Ω_t along a ray at the selected point.

Model modification consists in the replacement of the phase function Ψ_1 (5.20) by function Ψ_2, where the phase derivatives are considered as functions of the x and t variables rather than as constants.

$$\Psi_2 = \Psi_0 + k_x x + \frac{1}{2}K_x(x,t)x^2 + K_y(x,t)yx + \frac{1}{2}D(x,t)y^2 - \omega t +$$

$$+\Omega_x(x,t)xt + \Omega_y(x,t)yt - \frac{1}{2}\Omega_t(x,t)t^2. \qquad (5.22)$$

Values of the K_x, K_y, D, Ω_x, Ω_y, and Ω_t quantities at point X_0, Y_0, T_0 and the relationships between these quantities (5.8a), (5.8b), and (5.9a) remain the same. The expressions for the amplitude functions (5.18) and (5.19) also remain the same.

Substituting the new Anzatz $U = A\exp(i\Psi_2)$ into the initial wave equation and separating the real part, we obtain the equation of discrepancy between the exact and approximate solutions:

$$-A\left(k_x + \frac{1}{2}\frac{\partial K_x}{\partial x}x^2 + K_x x + \frac{\partial K_y}{\partial x}xy + K_y y + \frac{1}{2}\frac{\partial D}{\partial x}y^2 + \frac{\partial \Omega_x}{\partial x}xt + \Omega_x t +\right.$$

$$\left.+\frac{\partial \Omega_y}{\partial x}yt - \frac{1}{2}\frac{\partial \Omega_t}{\partial x}t^2\right)^2 - A(K_y x + Dy + \Omega_y t)^2 + A\frac{1}{c^2}\left(\varepsilon_0 + \frac{\partial \varepsilon_0}{\partial x}x +\right.$$

$$\left.+\frac{1}{2}\frac{\partial^2 \varepsilon_0}{\partial x^2}x^2 + \frac{\partial \varepsilon_0}{\partial y}y + \frac{1}{2}\frac{\partial^2 \varepsilon_0}{\partial y^2}y^2 + \frac{\partial^2 \varepsilon_0}{\partial x \partial y}xy\right)\left(-\omega + \frac{1}{2}\frac{\partial K_x}{\partial t}x^2 +\right.$$

$$+\frac{\partial K_y}{\partial t}xy + \frac{1}{2}\frac{\partial D}{\partial t}y^2 + \frac{\partial \Omega_x}{\partial t}xt + \Omega_x x + \frac{\partial \Omega_y}{\partial t}yt + \Omega_y y -$$

$$\left.-\frac{1}{2}\frac{\partial \Omega_t}{\partial t}t^2 - \Omega_t t\right)^2 - A\frac{1}{c^2}\left(\omega_M + \frac{\partial \omega_M}{\partial x}x + \frac{1}{2} + \frac{\partial^2 \omega_M}{\partial x^2}x^2 + \frac{\partial \omega_M}{\partial y}y +\right.$$

$$\left.+\frac{1}{2}\frac{\partial^2 \omega_M}{\partial y^2}y^2 + \frac{\partial^2 \omega_M}{\partial x \partial y}xy\right)^2 = 0. \qquad (5.23)$$

Here linear terms are already absent owing to (5.20). We now write the conditions of vanishing of the quadratic terms. The coefficient of y results in the following equation:

$$-k_x\frac{\partial D}{\partial x}-K_y^2-D^2-\frac{\varepsilon_0\omega}{c^2}\frac{\partial D}{\partial t}+\frac{\varepsilon_0}{c^2}\Omega_y^2-$$

$$-\frac{1}{2c^2}\frac{\partial^2\omega_M}{\partial y^2}+\frac{\omega^2}{2c^2}\frac{\partial^2\varepsilon_0}{\partial y^2}=0. \tag{5.24}$$

For the coefficient of the xt variables, we have

$$-k_x\frac{\partial\Omega_x}{\partial x}-K_x\Omega_x-K_y\Omega_y-\frac{\omega}{c^2}\frac{\partial\Omega_x}{\partial t}-\frac{\varepsilon_0}{c^2}\Omega_x\Omega_t=0. \tag{5.25}$$

For the coefficient of the yt variables, we have

$$-k_x\frac{\partial\Omega_y}{\partial x}-K_y\Omega_x-D\Omega_y-\frac{\varepsilon_0\omega}{c^2}\frac{\partial\Omega_y}{\partial t}-\frac{\varepsilon_0}{c^2}\Omega_y\Omega_t=0. \tag{5.26}$$

From Eqs. (5.24)–(5.26), we can rather simply obtain the total derivatives of Ω_x, Ω_y, and D along the ray using the relationship between the derivatives (5.6).

By acting in such a manner, we obtain

$$\frac{dD}{dt}=-\frac{c^2}{\varepsilon_0\omega}\left(\frac{\omega^2}{2c^2k_x}\frac{\partial\varepsilon_0}{\partial y}-\frac{1}{2c^2k_x}\frac{\partial\omega_M}{\partial y}\right)^2-\frac{1}{2\varepsilon_0\omega}\frac{\partial^2\omega_M}{\partial y^2}+$$

$$+\frac{\omega}{2\varepsilon_0}\frac{\partial^2\varepsilon_0}{\partial y^2}-\frac{\omega_M}{\omega(\varepsilon_0\omega^2-\omega_M)}\Omega_y^2+\frac{1}{\varepsilon_0\omega^2-\omega_M}\times$$

$$\times\left(\omega^2\frac{\partial\varepsilon_0}{\partial y}-\frac{\partial\omega_M}{\partial y}\right)\Omega_y-\frac{c^2}{\varepsilon_0\omega}D^2; \tag{5.24a}$$

$$\frac{d\Omega_x}{dt}=\frac{1}{k_x}\Omega_y^2+\left(\frac{1}{\varepsilon_0k_x}-\frac{c^2k_x}{\varepsilon_0\omega^2}\right)\Omega_x^2-\left(\frac{\omega}{2\varepsilon_0k_x}\frac{\partial\varepsilon_0}{\partial x}-\frac{1}{2\varepsilon_0\omega k_x}\frac{\partial\omega_M}{\partial x}\right)\Omega_x-$$

$$-\left(\frac{\omega}{2\varepsilon_0k_x}\frac{\partial\varepsilon_0}{\partial y}-\frac{1}{2\varepsilon_0\omega k_x}\frac{\partial\omega_M}{\partial y}\right)\Omega_y; \tag{5.25a}$$

$$\frac{d\Omega_y}{dt}=\left(\frac{1}{2\varepsilon_0\omega k_x}\frac{\partial\omega_M}{\partial y}-\frac{\omega}{2\varepsilon_0k_x}\frac{\partial\varepsilon_0}{\partial y}\right)\Omega_x-$$

$$-\frac{c^2}{\varepsilon_0\omega}\Omega_yD+\left(\frac{1}{k_x}-\frac{c^2k_x}{\varepsilon_0\omega^2}\right)\Omega_x\Omega_y. \tag{5.26a}$$

§ 7. Discussion of results

In this chapter we derived the closed set of ray equations for the media with an arbitrary time dispersion law and made sure that the conclusions that the dispersion refraction effect exists, drawn when we analyzed KGE, are valid for all types of dispersive media.

The method for obtaining the total derivatives of Ω_x, Ω_y, and D described above is beyond the scope of RO because this method takes into account not only the first and second derivatives but also the third derivative of the phase function. Here we take into account the effects of the second order of smallness $O(\chi^2)$, which could be classified as diffraction effects (refraction or RO effects are of the first order of smallness $O(\chi)$).

Generally speaking, the same procedures are performed in the classical RO, when the amplitude is determined from a ray tube divergence; in this case the third derivative of the phase function is also taken into account.

We called our last model a quasi-ray model since the solution for Ω_x, Ω_y, and D (as well as for the remaining wave field parameters in STRO) was reduced to total derivatives with respect to t along a ray.

Equations (5.21) completed with Eqs. (5.34a), (5.35a), and (5.36a) represent the closed set of ray equations for waves with frequency modulation. Monochromatic waves can be described using only Eqs. (5.21).

It follows from (5.25a) and (5.26a) that longitudinal and transverse frequency modulation can be transformed into each other [62, 72]. Transverse frequency modulation is transformed into longitudinal modulation even in a homogeneous medium, whereas inverse transformation can be observed in a medium with transverse inhomogeneity. This indicates that the effect of dispersion refraction can appear when wave packets with longitudinal modulation, emitted by ordinary antennas, propagate in an inhomogeneous medium with time dispersion.

It is clear that all results obtained in this chapter coincide with similar results obtained for KGE in the ray model and are valid for KGE in the quasi-ray model, if we set $\varepsilon_0 = 1$, $\omega_M = \omega_L^2$ in the generalized formulas.

The structures of the ray equations, obtained in this chapter for an arbitrary time dispersion law, are similar to the previously obtained structure of KGE because RO is an HF approximation and KGE is an asymptotic approximation for all types of time dispersion at increasing frequency or decreasing wavelength (see Chapter 1).