Chapter 3
Numerical description of the dispersion refraction effect using the non-stationary parabolic wave equation

§ 1. Synthesis of wave packets with transverse frequency modulation

Before considering the main subject-matter of this chapter, in the first section we will analyze the synthesis of wave packets with transverse modulation in a homogeneous medium.

Based on the most probable practical possibility of using the effect of dispersion refraction in the HF radio propagation in the ionosphere, we will consider the synthesis of packets as applied to this problem.

Let the emitter of electromagnetic waves be located on the Earth's surface. Assume that an inhomogeneous wave structure, which manifests itself in the dependence of emitted wave frequency ω on elevation angle φ, is formed in a homogeneous dispersion-free medium with a relative electric permittivity of $\varepsilon = 1$ and a relative magnetic permittivity of $\mu = 1$ until a wave reaches the ionosphere. If elevation angles are small (a usual situation during radio communication and over-the-horizon target location), the distance ρ from an emitter to the entrance into the ionosphere is not less than 500 km [48, 73].

Since ρ is much greater than the wavelength $\lambda = 10 - 100$ m and the possible physically realizable antenna height is $h = 10 - 100$ m, we will use the $\lambda/\rho \sim h/\rho \ll 1$ relationship as a small parameter of the problem.

The electric field vector **E** can be expressed in terms of the vector $\mathbf{A}_e$ and scalar φ_e potentials as [23, 54, 70]:

$$\mathbf{E} = -\nabla\phi_e - \frac{\partial \mathbf{A}_e}{\partial t}. \tag{3.1}$$

The scalar potential φ_e describes the static field component, which is substantial only in the near emitter zone (i.e., at distances comparable with the wavelength); consequently, this potential can be not considered because of λ/ρ smallness. In this case the electric field vector **E** is defined by the formula

$$\mathbf{E} = -\frac{\partial \mathbf{A}_e}{\partial t} \tag{3.2}$$

and is described by the equation for the vector potential

$$\nabla^2 \mathbf{A}_e - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_e}{\partial t^2} = \mu_0 \mathbf{J}, \tag{3.3}$$

where **J** is the extraneous current density.

For a homogeneous medium, the solution to Eq. (3.3) can be written in the integral form

$$\mathbf{A}_e(\mathbf{R}, t) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{R}, t - \mathbf{R}/c)}{\mathbf{R}} dV. \tag{3.4}$$

Here V is the region occupied by the extraneous currents, and **R** is the position vector of an observational point.

Taking into account the smallness of h/ρ, we can simplify the equation of retarded potentials (3.4)

$$\mathbf{A}_e(\rho, \phi, t) = \frac{\mu_0}{4\pi\rho}\int_0^h J(z, t - (\rho - z\sin\phi)/c)\, dz, \tag{3.5}$$

where z is the vertical coordinate coincident with the antenna aperture, ρ is the distance from the $z = 0$ point to an observational point, and φ is the elevation angle.

The condition of monochromaticity $J(z, t) = j(z)\exp(-i\omega t)$ makes it possible to obtain the field distribution $E(\rho, \varphi, t)$ in the far zone immediately from (3.2) and (3.5):

$$E(\rho, \phi, t) = \frac{i\omega\mu_0}{4\pi c\rho}\exp\left(i\frac{\omega}{c}\rho - i\omega t\right)\int_{-\infty}^{\infty} j(z)\exp\left(-i\frac{\omega}{c}\phi z\right)dz. \tag{3.6}$$

The $\sin\varphi$ function in (3.6) is replaced by the argument φ from the condition of elevation angle smallness, and the integration limits are extended to infinity, keeping in mind that the current is localized in the $h \ll \rho$ region. Equation (3.6) is well-known and is often used to synthesize atennas [22, 23, 25, 71]. Based on this equation, we consider the synthesis of the amplitude $j(z, t)$ of the current that forms the wave with transverse frequency modulation in the far zone

$$E(\rho, \phi, t) \sim \frac{1}{\rho}\exp\left\{i\left[\frac{\omega + \alpha\phi}{c}\rho - (\omega + \alpha\phi)t\right]\right\} \tag{3.7}$$

at distance ρ from an emitter in a rather narrow range of angles $-\varphi_1 < \varphi < \varphi_1$. Assume that the field strength is zero outside this range; i.e., we will synthesize an ideal emitter without secondary lobes. At $\alpha\varphi_1 \ll \omega$, Eq. (3.7) describes a quasi-monochromatic wave with weak transverse frequency modulation (α is the coefficient of angular frequency modulation), which makes it possible to use Eq. (3.6) in synthesis.

From (3.6) and (3.7), we obtain the equation for the modulating function E_M:

$$E_M(\rho,\phi,t) = \exp\left(i\frac{\alpha\phi}{c}\rho - i\alpha\phi t\right) = \int_{-\infty}^{\infty} j(z,t)\exp(-i\frac{\omega}{c}\phi z)dz. \quad (3.8)$$

From the obtained equation it follows that the modulating function and the current amplitude are connected by the Fourier transform, which makes it possible to express the amplitude in terms of the modulating function:

$$j(z,t) = \frac{c}{2\pi\omega}\int_{-\phi_1}^{\phi_1} \exp\left(i\frac{\alpha\phi}{c}\rho - i\alpha\phi t\right)\exp\left(i\frac{\omega}{c}\phi z\right)d\phi \quad (3.9)$$

or

$$j(z,t) = \frac{c}{\pi\omega}\frac{\sin[((\omega/c)z - \alpha t + (\alpha/c)\rho)\phi_1]}{(\omega/c)z - \alpha t + (\alpha/c)\rho} \quad (3.10)$$

after integration.

Equation (3.10) describes the current amplitude form in the antenna, which is a certain spatial envelope of the sin *z/z* type, "running" along the *z* axis at a velocity of $V_z = \alpha c/\omega$.

In the far zone, the antenna current $J(z,i) = j(z, t)\exp(-i\omega t)$ forms a weakly inhomogeneous wave with the frequency modulation angular coefficient α in the $-\varphi_1 < \varphi < \varphi_1$ range of angles.

Spatial envelop (3.10) is defined only by the selected angular intensity curve and can be arbitrary, e.g., δ-shaped. In the latter case, an emitter is equivalent to a monochromatic point source of frequency ω moving at velocity V_z. It is clear that, because of the Doppler effect, the maximal offset frequency will be observed ahead of the source; the minimal frequency – behind the source; and the maximal angular coefficient of frequency modulation – in the direction perpendicular to the velocity vector. S.M. Rytov indicated that a moving point source is the simplest example of a wave emitter with transverse frequency modulation.

In the following chapters, we will show that the transformation of longitudinal modulation into transverse modulation can be used to form a wave with transverse frequency modulation in an inhomogeneous medium. Such a method makes it possible to apply usual antennas but has one disadvantage, i.e., the dependence on the specific structure of an inhomogeneous medium.

§ 2. Non-stationary parabolic wave equation

Numerical methods for solving wave problems sometimes make it possible to avoid complex analytical computations and to rapidly obtain the necessary result. Unfortunately, it is impossible to numerically solve the complete wave equation for real media (e.g., KGE for HF radio propagation in the ionosphere) even at the current level of computing technologies. Therefore, it is reasonable to use "abridged" parabolic wave equations, which are small-angle approximations of the corresponding "complete" equations, in particular, for the ionosphere because, first, small-angle approximations are satisfied in these problems, and, second, these equations can be solved numerically.

We now consider the derivation of the two-dimensional non-stationary parabolic equation (NPE) for KGE (1.7) in orthogonal coordinates x, y [58, 93].

We represent the wave field in the form

$$U = A(x,y,t)\exp(ik_0 x - i\omega_0 t). \tag{3.11}$$

Upon substituting (3.11) into (1.7) and rejecting the second derivatives with respect to x and t, we obtain the non-stationary parabolic equation

$$2ik_0\frac{\partial A}{\partial x} + 2i\frac{\omega_0}{c^2}\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial y^2} - \frac{\omega_L^2}{c^2}A = 0. \tag{3.12}$$

Here A is a "slow" complex-valued function of the x, y, and t variables; ω_0 is the wave packet central frequency; and $k_0 = \omega_0/c$ is the wavenumber.

If a "slow" complex amplitude $A(x,y,t)$ is guaranteed during the solution of the parabolic equation, differential equation (3.12) can certainly be approximated by the finite-difference equation on the time and spatial grid with large spatial and temporal steps exceeding the wavelength and oscillation period. As a result, the considered problem can be solved numerically.

The condition of parabolic equation applicability is based on the "smoothness" and "slowness" of amplitude A, which makes it possible to reject the second derivatives as compared to the first derivatives; i.e.,

$$2k_0\frac{\partial A}{\partial x} \gg \frac{\partial^2 A}{\partial x^2},$$

$$2\omega_0\frac{\partial A}{\partial t} \gg \frac{\partial^2 A}{\partial t^2}.$$

This condition is realized when a wave propagates mostly along the x axis with an insignificant angular divergence.

Evidently, the dispersion relations for the parabolic equation and original KGE coincide incompletely.

Let us rewrite the dispersion relation for KGE (1.10) in the form

$$(k_0 + K_x)^2 + K_y^2 = \frac{(\omega_0 + \Omega)^2}{c^2} - \frac{\omega_L^2}{c}. \tag{3.13}$$

Here we divided the wavenumber k and frequency ω into two parts, where k_0 and ω_0 belong to plain wave (3.11), and K_x, K_y, and Ω enter into the amplitude function

$$A = A_0 \exp(iK_x x + iK_y y - i\Omega t), \tag{3.14}$$

i.e., $U = A_0 \exp\{i(k_0 + K_x)x + iK_y y - i(\omega_0 + \Omega)t\}$.

By substituting (3.14) into (3.12), we obtain the dispersion equation for NPE

$$2k_0 K_x + K_y^2 = \frac{2\omega_0 \Omega}{c^2} - \frac{\omega_L^2}{c^2}. \tag{3.15}$$

If we reject the K_x^2 and Ω^2 terms in the $(k_0 + K_x)^2$ and $(\omega_0 + \Omega)^2$ expressions, we can obtain Eq. (3.15) from (3.13).

Consequently, the dispersion equation (3.15) approximates the exact dispersion law for KGE, if $k_0 \gg K_x$ and $\omega_0 \gg \Omega$.

§ 3. Finite-difference equation for NPE

We will use the following finite-difference approximation of the parabolic equation (3.12)

$$A_{x,y}^{t+1} = A_{x,y}^{t-1} - \frac{c\Delta t}{\Delta x}\left(A_{x+1,y}^{t} - A_{x-1,y}^{t}\right) + \frac{ic\Delta t}{k_0(\Delta y)^2}(A_{x,y+1}^{t} -$$
$$- 2A_{x,y}^{t} + A_{x,y-1}^{t}) - \frac{i\omega_L^2 \Delta t}{k_0 c} A_{x,y}^{t}. \tag{3.16}$$

This is an explicit conditionally stable scheme of the second order of accuracy along all the coordinates [102]. Here A is the digital complex-valued mesh amplitude function, where the superscripts and subscripts mark the temporal layers and the spatial mesh node numbers.

The values of Δt, Δx, and Δy are the temporal and spatial steps along the t, x, and y axes, respectively.

Below, we will calculate the stability of the finite-difference equation (3.16).

Assume that

$$A = A_0 \exp(ik_x + ik_y) \tag{3.17}$$

is the wave Fourier mode.

We introduce the multiplier of transition g between temporal layers, i.e.,

$$A_{x,y}^{t+1} = gA_{x,y}^{t}, \qquad A_{x,y}^{t-1} = \frac{A_{x,y}^{t}}{g}.$$

Substituting (3.17) into (3.16) and reducing the first equation by $A = A_0 \exp(ik_x + ik_y)$, we obtain

$$g - \frac{1}{g} + \frac{c\Delta t}{\Delta x}[\exp(ik_x\Delta x) - \exp(-ik_x\Delta x)] - \frac{ic\Delta t}{k_0 \Delta y^2} \times$$

$$\times [\exp(ik_y\Delta y) - 2 + \exp(-ik_y\Delta y)] + \frac{2\omega_L^2 \Delta t}{k_0 c} = 0.$$

Taking into account that

$$2\cos\alpha = e^{i\alpha} + e^{-i\alpha}; \qquad i2\sin\alpha = e^{i\alpha} - e^{-i\alpha}$$

we have

$$g - \frac{1}{g} + \frac{c\Delta t}{\Delta x}[2i\sin(k_x\Delta x)] - \frac{2ic\Delta t}{k_0(\Delta y)^2} \times$$

$$\times [\cos(k_y\Delta y) - 1] + \frac{i\omega_L^2 \Delta t}{k_0 c} = 0,$$

or

$$g^2 + ibg - 1 = 0, \tag{3.18}$$

where

$$b = \frac{c\Delta t}{\Delta x}[2i\sin(k_x\Delta x)] - \frac{2ic\Delta t}{k_0(\Delta y)^2}[\cos(k_y\Delta y) - 1] + \frac{i\omega_L^2 \Delta t}{k_0 c}.$$

The roots of Eq. (3.18) have the form

$$g_{1,2} = -\frac{ib}{2} \pm \sqrt{-\frac{b^2}{4} - 1}.$$

Finite-difference scheme (3.16) is stable if

$$|g| \le 1.$$

The condition for b follows from this inequality

$$-2 \le b \le 2.$$

Since the latter inequality should be satisfied at any harmonic function arguments, we come to the stability condition of a difference approximation for temporal and spatial steps of the mesh function

$$\frac{1}{\Delta t} \geqslant \frac{c}{\Delta x} + \frac{2c}{k_0(\Delta y)^2} + \frac{\omega_L^2}{2k_0 c}.$$

A finite-difference approximation (3.16) of the parabolic equation (3.12) can be used only if the dispersion equation of this approximation corresponds to the dispersion equation (3.15).

The dispersion equations for the x and y axes are different for the finite-difference scheme (3.16) and NPE; therefore, we will consider these equations separately.

We consider the Fourier-mode for the x axis

$$A = A_0 \exp(iK_x x - i\Omega t). \tag{3.19}$$

We use the central point of the pattern as an origin. Substituting (3.19) into (3.16), we have

$$\exp(i\Omega\Delta t) - \exp(-i\Omega\Delta t) - \frac{c\Delta t}{\Delta x}[\exp(iK_x\Delta x) -$$

$$- \exp(-iK_x\Delta x)] - \frac{\omega_L^2\Delta t}{k_0 c} = 0$$

or

$$2\frac{c\Delta t}{\Delta x}\sin(K_x\Delta x) = 2\sin(\Omega\Delta t) - \frac{\omega_L^2\Delta t}{k_0 c}.$$

At small $K_x\Delta x$ and $\Omega\Delta t$ values, this dispersion equation approaches Eq. (3.15) for the axial coordinate:

$$K_x = \frac{\Omega}{c} - \frac{\omega_L^2}{2k_0 c}.$$

The approximation condition can be written as

$$\Delta t \ll \frac{2\pi}{\Omega}, \quad \Delta x \ll \frac{2\pi}{K_x}.$$

We now consider the Fourier-mode for the y axis

$$A = A_0 \exp(iK_y y - i\Omega t). \tag{3.20}$$

Substituting (3.20) into (3.16), we have

$$\exp(i\Omega\Delta t) - \exp(-i\Omega\Delta t) - \frac{ic\Delta t}{k_0\Delta y}[\exp(iK_y\Delta y) -$$

$$- 2 + \exp(-iK_y\Delta y)] - \frac{i\omega_L^2\Delta t}{k_0 c} = 0.$$

Let us rewrite this equation in the other form

$$\frac{c\Delta t}{k_0\Delta y}[\cos(K_y\Delta y) - 1] = 2\sin(\Omega\Delta t) - \frac{\omega_L^2\Delta t}{k_0 c}.$$

We obtained the dispersion equation for the transverse coordinate of the finite-difference equation (3.16). At small $\Delta\Omega t$ and $K_y\Delta y$ values, this dispersion equation approaches the dispersion equation (3.15) for the transverse coordinate

$$K_y^2 = \frac{2k_0\Omega}{c} - \frac{\omega_L^2}{c^2}.$$

Here, the approximation condition has a similar form

$$\Delta t \ll \frac{2\pi}{\Omega}; \quad \Delta y \ll \frac{2\pi}{K_y}.$$

§ 4. Results of numerical computations

The finite-difference equation (3.16) was numerically solved in the rectangular spatial region $[0, L_x]$, $[0, L_y]$ along the x and y axes. We solved the wave problem under zero initial conditions $A = 0$ and $dA/dt = 0$ at $t = 0$ for a homogeneous medium. Zero conditions were specified at the $y = 0$, $y = L_y$, and $x = L_x$ boundaries. At the $x = 0$ boundary, the wave amplitude was specified as follows:

$$A = A_0 \exp\left\{-\frac{(t-t_0)^2}{\sigma_t^2} - \frac{(y-y_0)^2}{\sigma_y^2} + i2\pi F_m (t-t_0)(y-y_0)\right\}. \quad (3.21)$$

Formula (3.21) describes a frequency-modulated wave packet with the Gaussian envelop with duration σ_t along the time coordinate t and width σ_y along the transverse coordinate y. The value of transverse frequency modulation is specified by the modulation index F_m. The following parameters were selected during the calculations:

Central frequency $f_0 = \omega_0/2\pi$	5 MHz
Period σ_t	5 μs
Width σ_y	1.2 km
t_0	15 μs
y_0	4 km
L_x boundary	10 km
L_y boundary	8 km
Δx step	0.04 km
Δy step	0.05 km
Modulation index F_m	0.08 MHz/km

Figures 10 and 11 show the spatial distribution of the amplitude module $|A|$ in free space without dispersion (i.e., at $\omega_L = 0$). Figures 10 and 11 show the wave packets at $t = 25$ and 35 mks, respectively.

Figures 12–15 show a similar field distribution for a homogeneous medium with time dispersion at $f_L = \omega_L/2\pi = 3.5$ MHz. Figures 12–15 correspond to observation times of $t = 25$, 35, 45, and 55 mks, respectively.

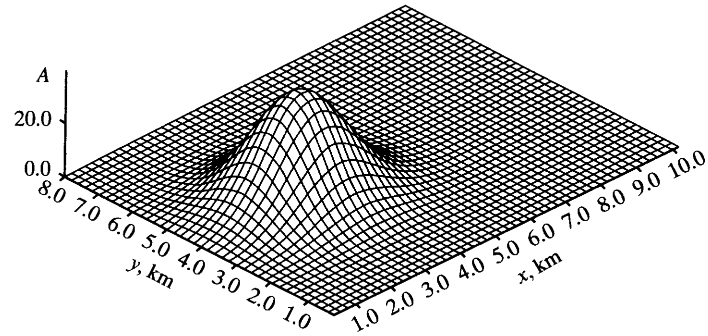

Fig. 10. Wave packet envelope in dispersion-free media at instant $t = 25$ mks

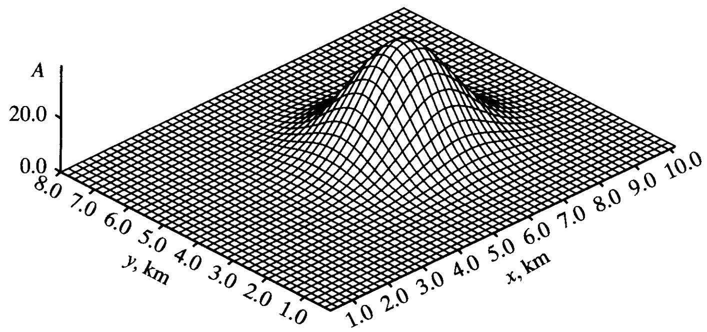

Fig. 11. Wave packet envelope in dispersion-free media at instant $t = 35$ mks

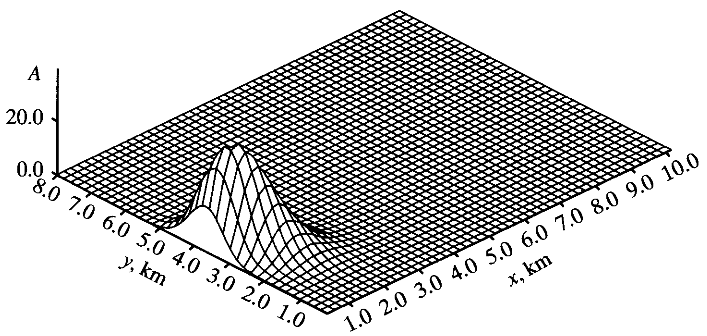

Fig. 12. Wave packet envelope in dispersion media at instant $t = 25$ mks

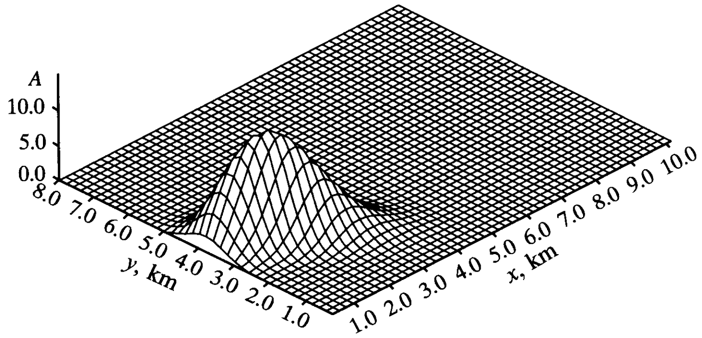

Fig. 13. Wave packet envelope in dispersion media at instant $t = 35$ mks

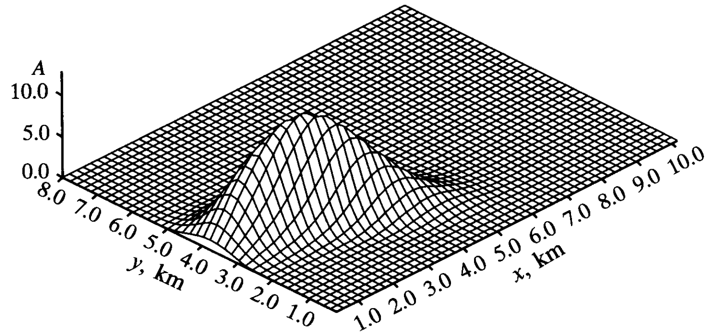

Fig. 14. Wave packet envelope in dispersion media at instant $t = 45$ mks

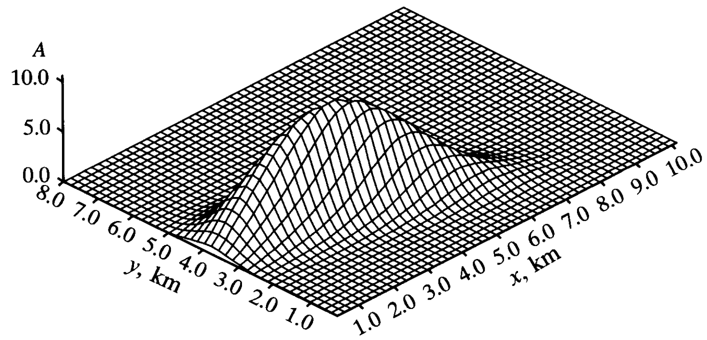

Fig. 15. Wave packet envelope in dispersion media at instant $t = 55$ mks

§ 5. Discussion of results

It can be observed in Figs. 10 and 11 that a two-dimensional wave packet with transverse frequency modulation in a homogeneous dispersion-free medium propagates linearly, slightly spreading due to diffraction.

Figures 12–15 clearly demonstrate that a homogeneous dispersive medium affects the wave packet form. First, an axial packet extension, which is often called dispersive distortions, takes place. Second, a transverse packet shift toward the right-hand boundary, caused by the dispersion refraction effect, is clearly defined.

Both these effects are of the same nature since they are characterized by the nonlinear dependence of frequency ω on wavenumber k in the dispersion equation. Dispersion refraction can be considered among transverse dispersive distortions of a wave packet.

The first effect has a more general nature since this effect can be observed in a one-dimensional medium. Transverse distortions can be observed only in $2D$ and $3D$ media and have not been studied as thoroughly as longitudinal distortions.